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Derivation of PID parameters using the moment method

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Consider the following process transfer function:

$$G_P(s) = K_{PR}\frac{1+b_1s+b_2s^2+\cdots+b_ms^m}{1+a_1s+a_2s^2+\cdots+a_ns^n}e^{-sT_{delay}} \quad , \tag{1}$$

and the following PID controller:

$$U = K\left(1+\frac{1}{sT_i}+\frac{sKT_d}{1+sT_f}\right)(W-Y) = G_C(s)(W-Y), \tag{2}$$

The following open-loop system transfer function is obtained from (1) and (2):

$$G_C(s)G_P(s) = \frac{d_0+d_1s+d_2s^2+d_3s^3+\ldots+d_qs^q}{c_0s+c_1s^2+c_2s^3+c_3s^4+\ldots+c_ps^p}e^{-sT_{delay}}, \tag{3}$$

where $G_C(s)G_P(s)$ is a strictly proper transfer function. Constants c_i and d_i in (7) can be calculated by inserting (1) and (2) into (3):

$$\begin{gathered}
c_0 = T_i \\
c_1 = a_1T_i \\
c_2 = a_2T_i \\
c_3 = a_3T_i \\
c_4 = a_4T_i \\
c_5 = a_5T_i \\
\vdots \\
d_0 = KK_{PR} \\
d_1 = KK_{PR}(b_1+T_i) \\
d_2 = KK_{PR}(b_2+b_1T_i+T_dT_i) \\
d_3 = KK_{PR}(b_3+b_2T_i+b_1T_dT_i) \\
d_4 = KK_{PR}(b_4+b_3T_i+b_2T_dT_i) \\
d_5 = KK_{PR}(b_5+b_4T_i+b_3T_dT_i) \\
\vdots
\end{gathered} \tag{4}$$

According to the derivation given in Hanus (1975), for the systems without pure time delay, the following expressions should be fulfilled to achieve the Magnitude Optimum (Åström, (1995), Umland and Safiuddin, (1990)) tuning:

$$\sum_{i=0}^{2n+1}(-1)^i d_ic_{2n+1-i} = \frac{1}{2}\sum_{i=0}^{2n}(-1)^i c_ic_{2n-i} \tag{5}$$

To find three PID parameters, first three equations in (5) must hold:

$$d_1c_0 - d_0c_1 + \frac{c_0^{\ 2}}{2} = 0 \tag{6}$$

$$-d_1c_2 - d_3c_0 + d_0c_3 + d_2c_1 - c_0c_2 + \frac{c_1^{\;2}}{2} = 0 \tag{7}$$

$$d_1c_4 + d_3c_2 + d_5c_0 - d_0c_5 - d_2c_3 - d_4c_1 + c_0c_4 - c_1c_3 + \frac{c_2^{\;2}}{2} = 0 \tag{8}$$

When inserting (4) into (6) to (8) the following expressions are obtained, respectively:

$$K = -\frac{T_i}{2K_{PR}\left(T_i - a_1 + b_1\right)} \tag{9}$$

$$T_i = -\frac{2KK_{PR}\left(a_1b_2 - a_2b_1 + a_3 - b_3\right)}{2KK_{PR}\left(T_d\left(a_1 - b_1\right) + a_1b_1 - a_2 - b_2\right) + a_1^{\;2} - 2a_2} \tag{10}$$

$$\begin{aligned} T_d = &-\frac{2KK_{PR}\left(T_i\left(a_1b_3 - a_2b_2 + a_3b_1 - a_4 - b_4\right) + a_1b_4 - a_2b_3 + a_3b_2 - a_4b_1 + a_5 - b_5\right)}{2KK_{PR}T_i\left(a_1b_2 - a_2b_1 + a_3 - b_3\right)} + \\ &+ \frac{T_i\left(2a_1a_3 - a_2^{\;2} - 2a_4\right)}{2KK_{PR}T_i\left(a_1b_2 - a_2b_1 + a_3 - b_3\right)} \end{aligned} \tag{11}$$

The PID controller parameters can be expressed only by the process parameters a_1 to a_5, b_1 to b_5 by solving equations (9) to (11). When inserting (9) into (10), the following expressions can be obtained:

$$K = -\frac{a_1^{\;3} - a_1^{\;2}b_1 + a_1\left(b_2 - 2a_2\right) + a_2b_1 + a_3 - b_3}{2K_{PR}\left(T_d\left(a_1^{\;2} - 2a_1b_1 + b_1^{\;2}\right) + a_1^{\;2}b_1 - a_1\left(a_2 + b_1^{\;2}\right) + a_3 + b_1b_2 - b_3\right)} \tag{12}$$

$$T_i = -\frac{a_1^{\;3} - a_1^{\;2}b_1 + a_1\left(b_2 - 2a_2\right) + a_2b_1 + a_3 - b_3}{T_d\left(a_1 - b_1\right) - a_1^{\;2} + a_1b_1 + a_2 - b_2} \tag{13}$$

The expression can be simplified by using the expressions for areas, which can be obtained from the process step response (Vrančić, 1995; Vrančić et al., 1996). Following Strejc (1959), the following five areas can also be expressed by the unknown parameters:

$$A_1 = \int_0^\infty \left(K_{PR} - y(t)\right)dt = \lim_{s\to 0}\frac{1}{s}\left(K_{PR} - G_P(s)\right) = K_{PR}\left(a_1 - b_1 + T_{delay}\right) \tag{14}$$

$$\begin{aligned} A_2 = y_2(\infty) &= \int_0^\infty \left(A_1 - y_1(t)\right)dt = \lim_{s\to 0}\frac{1}{s}\left(A_1 - \frac{1}{s}\left(K_{PR} - G_P(s)\right)\right) = \\ &= K_{PR}\left[b_2 - a_2 + A_1a_1 - T_{delay}b_1 + \frac{T_{delay}^{\;2}}{2}\right] \end{aligned} \tag{15}$$

$$A_3 = y_3(\infty) = \int_0^{\infty}\left(A_2 - y_2(t)\right)dt = \lim_{s\to 0}\frac{1}{s}\left(A_2 - \frac{1}{s}\left(A_1 - \frac{1}{s}\left(K_{PR} - G_P(s)\right)\right)\right) =$$
$$= K_{PR}\left[a_3 - b_3 + A_2a_1 - A_1a_2 + T_{delay}b_2 - \frac{T_{delay}{}^2 b_1}{2} + \frac{T_{delay}{}^3}{6}\right] \tag{16}$$

$$A_4 = \int_0^{\infty}\left(A_3 - y_3(t)\right)dt = K_{PR}\left[b_4 - a_4 + A_3a_1 - A_2a_2 + A_1a_3 - T_{delay}b_3 + \frac{T_{delay}{}^2 b_2}{2} - \frac{T_{delay}{}^3 b_1}{3!} + \frac{T_{delay}{}^4}{4!}\right] \tag{17}$$

$$A_5 = \int_0^{\infty}\left(A_4 - y_4(t)\right)dt =$$
$$= K_{PR}\left[a_5 - b_5 + A_4a_1 - A_3a_2 + A_2a_3 - A_1a_4 + T_{delay}b_4 - \frac{T_{delay}{}^2 b_3}{2} + \frac{T_{delay}{}^3 b_2}{3!} - \frac{T_{delay}{}^4 b_1}{4!} + \frac{T_{delay}{}^5}{5!}\right], \tag{18}$$

where *y(t)* denotes the process open-loop step response, and

$$y_1(t) = \int_0^{t}\left(K_{PR} - y(\tau)\right)d\tau \tag{19}$$

$$y_2(t) = \int_0^{t}\left(A_1 - y_1(\tau)\right)d\tau \ . \tag{20}$$

$$y_3(t) = \int_0^{t}\left(A_2 - y_2(\tau)\right)d\tau \tag{21}$$

$$y_4(t) = \int_0^{t}\left(A_3 - y_3(\tau)\right)d\tau \ . \tag{22}$$

When inserting the areas (14) to (18), obtained from the step response (see e.g. Vrančić, (1995), Isermann, (1971) or Rake, (1987), into equations (12) and (13), and considering T_{delay}=0, the following result is obtained:

$$K = \frac{A_3K_{PR}}{2K_{PR}\left(A_1A_2 - A_3K_{PR} - T_d A_1^2\right)} \tag{23}$$

$$T_i = \frac{A_3}{A_2 - T_d A_1} \tag{24}$$

Similarly, we can obtain the value of T_d by inserting (23) and (24) into (11):

$$T_d = \frac{A_3\left(a_1(2a_3-b_3)-a_2^{\ 2}+a_2b_2-a_3b_1-a_4+b_4\right)+\ldots}{A_3\left(a_1b_2-a_2b_1+a_3-b_3\right)+\ldots}$$

$$\frac{\ldots+A_2\left(-2a_1^{\ 2}a_3+a_1\left(a_2^{\ 2}+2a_3b_1+2a_4-b_4\right)-a_2^{\ 2}b_1+a_2b_3-a_3b_2-a_4b_1-a_5+b_5\right)}{\ldots+A_1\left(-2a_1^{\ 2}a_3+a_1\left(a_2^{\ 2}+2a_3b_1+2a_4-b_4\right)-a_2^{\ 2}b_1+a_2b_3-a_3b_2-a_4b_1-a_5+b_5\right)} \quad (25)$$

By some mathematical exercise it can be shown that:

$$\begin{aligned}&A_3\left(a_1(2a_3-b_3)-a_2^{\ 2}+a_2b_2-a_3b_1-a_4+b_4\right)+\\&+A_2\left(-2a_1^{\ 2}a_3+a_1\left(a_2^{\ 2}+2a_3b_1+2a_4-b_4\right)-a_2^{\ 2}b_1+a_2b_3-a_3b_2-a_4b_1-a_5+b_5\right)=\\&=A_3A_4-A_2A_5\end{aligned} \quad (26)$$

Similarly it can be shown that the following expression holds for the denominator of T_d:

$$\begin{aligned}&A_3\left(a_1b_2-a_2b_1+a_3-b_3\right)+\\&+A_1\left(-2a_1^{\ 2}a_3+a_1\left(a_2^{\ 2}+2a_3b_1+2a_4-b_4\right)-a_2^{\ 2}b_1+a_2b_3-a_3b_2-a_4b_1-a_5+b_5\right)=\\&=A_3^{\ 2}-A_1A_5\end{aligned} \quad (27)$$

Therefore, the parameter T_d can be expressed as:

$$T_d = \frac{A_3A_4-A_2A_5}{A_3A_3-A_1A_5} \quad (28)$$

Let us define factor α_D as:

$$\alpha_D = \frac{A_1\left(A_2A_3-A_1A_4\right)}{K_{PR}\left(A_3^{\ 2}-A_1A_5\right)}-1 \quad (29)$$

Then, (23) and (24) can be rewritten into:

$$K = \frac{0.5}{K_{PR}\alpha_D} \quad (30)$$

$$T_i = \frac{A_1}{K_{PR}\left(1+\alpha_D\right)} \quad (31)$$

Note that the same result (controller parameters expressed by the areas) holds also when considering pure time delay ($T_{delay}\neq 0$).

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