Survey of Gain-Scheduling Analysis & Design

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Abstract
The gain-scheduling approach is perhaps one of the most popular nonlinear control design approaches which has been widely and successfully applied in fields ranging from aerospace to process control. Despite the wide application of gain-scheduling controllers and a diverse academic literature relating to gain-scheduling extending back nearly thirty years, there is a notable lack of a formal review of the literature. Moreover, whilst much of the classical gain-scheduling theory originates from the 1960s, there has recently been a considerable increase in interest in gain-scheduling in the literature with many new results obtained. An extended review of the gain-scheduling literature therefore seems both timely and appropriate. The scope of this paper includes the main theoretical results and design procedures relating to continuous gain-scheduling (in the sense of decomposition of nonlinear design into linear sub-problems) control with the aim of providing both a critical overview and a useful entry point into the relevant literature.

1. Introduction

Gain-scheduling is perhaps one of the most popular approaches to nonlinear control design and has been widely and successfully applied in fields ranging from aerospace to process control. Although a wide variety of control methods are often described as "gain-scheduling" approaches, these are usually linked by a divide and conquer type of design procedure whereby the nonlinear control design task is decomposed into a number of linear sub-problems. This divide and conquer approach is the source of much of the popularity of gain-scheduling methods since it enables well established linear design methods to be applied to nonlinear problems. (Whilst the analysis and design of nonlinear systems remains relatively difficult, techniques for the analysis and design of linear time-invariant systems are rather better developed). However, it is also emphasised that the benefits of continuity with linear methods often extend beyond purely technical considerations; for example, safety certification requirements are often based on linear methods and the development of new certification procedures using nonlinear approaches may well be prohibitive. Of course, the question must be asked as to whether the basic premise of such design approaches is in fact reasonable; that is, whether a wide class of nonlinear design tasks can genuinely be decomposed into linear sub-problems. While few results are available which relate directly to this fundamental issue, and it is well known that certain classes of problem present greater difficulty than others for gain-scheduling methods, the general usefulness of such methods is nevertheless well established both in practice and from a theoretical viewpoint.

Despite the wide application of gain-scheduling controllers and a diverse academic literature relating to gain-scheduling extending back nearly thirty years, there is a notable lack of a formal review of the literature. Moreover, whilst much of the classical gain-scheduling theory originates from the 1960s, there has recently been a considerable increase in interest in gain-scheduling in the literature with many new results obtained. A review of the gain-scheduling literature therefore seems both timely and appropriate. It is, unfortunately, impossible to cover the great wealth of gain-scheduling literature within the available space and the scope of this paper is thus necessarily limited. Firstly, no attempt is made to review the vast literature detailing specific applications of gain-scheduling methods. Secondly, in order to retain a reasonable focus to the paper, consideration is confined to methods based on the decomposition of the nonlinear design task into *linear* sub-problems; accordingly, the many interesting approaches based on decomposing the design task into simpler nonlinear sub-problems (including those involving decomposition into affine sub-problems) are not discussed. Thirdly, attention is restricted to *continuous* scheduling methods and no attempt is made to review the very extensive literature on hybrid/switched systems. The scope of the paper is thus restricted to the main theoretical results and design procedures relating to continuous gain-scheduling (in the sense of decomposition of nonlinear design into linear sub-problems) control and the aim is to provide a critical overview and useful entry point into the relevant literature. The subject matter covered clearly remains considerable and it has sometimes been difficult to achieve a satisfactory balance between the requirement to provide a concise critical overview of the field while covering the subject in reasonable detail. In addition, although substantial effort has been expended in striving to present as balanced perspective as possible on alternative methodologies, the reader should be aware that some degree of subjective judgement is surely inevitable. Any such deficiencies do not, of course, necessarily reflect the opinions of others.

The paper is organised as follows. The theoretical results relating the dynamic characteristics of a nonlinear system to those of a family of linear systems are reviewed in section 2. The classical gain-scheduling design procedure is discussed in section 3 followed by a number of recent divide and conquer approaches which attempt to address a number of deficiencies of classical methods. LPV gain-scheduling approaches, which have recently been the subject of considerable research activity but are less strongly based on divide and conquer ideas, are reviewed in section 4 and the outlook is briefly discussed in section 5. The notation used is standard (see Appendix A).

2. Linearisation theory

Gain-scheduling design typically employs a divide and conquer approach whereby the nonlinear design task is decomposed into a number of linear sub-tasks. Such a decomposition depends on establishing a relationship between a nonlinear system and a family of linear systems. The main theoretical results which, for a broad class of nonlinear systems, relate the dynamic characteristics of a member of the class to those of an associated family of linear systems are reviewed in this section. These results fall into two main sub-classes. First, stability results which establish a relationship between the stability of a nonlinear system and the stability of an associated linear system. Second, approximation results which establish a direct relationship between the solution to a nonlinear system and the solution to associated linear systems. It is important to distinguish between these classes of result. The former are typically much more limited than the latter, being confined to specifying conditions under which boundedness of the solution to a particular linear system implies boundedness of the solution to the nonlinear system for an appropriate class of inputs and initial conditions. Notice that under such conditions the solutions are bounded but may otherwise be quite dissimilar. Reflecting this distinction, the discussion in the following sections often separately considers results relating both to stability and approximation.

The section is organised as follows. Perhaps the most widespread approach for associating a linear system with a nonlinear one, namely series expansion linearisation theory, is first reviewed The literature relating to series expansion linearisations is, of course, extensive yet, unfortunately, also very fragmented and it is necessary to consolidate many separate results in writing this review. The approach taken here is, therefore, to provide an overview of the available body of theory in section 2.1 while referring the reader to Appendix B for detailed references. The series expansion linearisation is only valid in the vicinity of a specific trajectory or equilibrium point, and so there is considerable incentive to develop techniques which relax this restriction. Approaches which aim to increase the allowable operating envelope by utilising a family of linearisations (rather than just a single linearisation) are reviewed in sections 2.2-2.3.

2.1 Series expansion linearisation about a single trajectory or equilibrium point[1]

The series expansion linearisation of a nonlinear system is well known. Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}), \quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{1}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$. Let ($\tilde{\mathbf{x}}$ (t), $\tilde{\mathbf{r}}$ (t), $\tilde{\mathbf{y}}$ (t)) denote a specific trajectory of the nonlinear system (the trajectory could simply be an equilibrium operating point in which case $\tilde{\mathbf{x}}$ is constant). Neglecting higher-order terms, it follows from series expansion theory that the nonlinear system, (1), may be approximated, locally to the trajectory, ($\tilde{\mathbf{x}}$ (t), $\tilde{\mathbf{r}}$ (t), $\tilde{\mathbf{y}}$ (t)), by the linear time-varying system

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} \tag{2}$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\mathbf{r} \tag{3}$$

where

$$\delta\mathbf{r} = \mathbf{r} - \tilde{\mathbf{r}}, \ \mathbf{y} = \delta\mathbf{y} + \tilde{\mathbf{y}}, \ \delta\mathbf{x} = \mathbf{x} - \tilde{\mathbf{x}} \tag{4}$$

Clearly, the nonlinear system, (1), is stable relative to the trajectory ($\tilde{\mathbf{x}}$ (t), $\tilde{\mathbf{r}}$ (t), $\tilde{\mathbf{y}}$ (t)) provided the linear time-varying dynamics (2)-(3) are robustly stable with respect to the approximation error involved in truncating the series expansion. In fact, is turns out that nonlinear system, (1), is locally BIBO stable if and only if the linear system (2)-(3) is exponentially stable when $\delta\mathbf{r}$ is zero (*i.e.* the unforced case) (see, for example, Khalil 1992 p184, Vidyasagar & Vannelli 1982).

In the special case when the system, (2)-(3), is linear time-invariant, simple necessary and sufficient conditions for its stability are well-known (see, for example, Vidyasagar 1993). However, in the time-varying case, the stability analysis is, in general, not so straightforward. In the context of gain-scheduling, frozen-time theory is widely employed to establish stability conditions for linear time-varying systems. Specifically, it can be shown(see, for example, Desoer 1969, Ilchmann *et al.* 1987) that the stability of the linear time-varying system, (2)-(3), is guaranteed provided that the time variation of $\nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$ is sufficiently slow in some appropriate sense (for example, that $\sup_{t\geq 0} |d/dt(\nabla_{\mathbf{x}} F(\tilde{\mathbf{x}}, \tilde{\mathbf{r}}))|$ is sufficiently small). Although classical frozen-time results mainly relate to nominal stability, it can also be shown that, provided the rate of variation is sufficiently

[1] See also Appendix B

slow, the linear time-varying system (2)-(3) inherits the worst-case stability robustness of the family of frozen-time linear time-invariant systems $\delta\dot{\hat{\mathbf{x}}} = \mathbf{A}_\tau\delta\hat{\mathbf{x}}$ where $\mathbf{A}_\tau$ denotes the value of $\nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})$ at time τ (Barman 1973, Shamma & Athans 1991, Leith & Leithead 1998b). Frozen-time theory is generally conservative in that it only establishes sufficient conditions for stability. In addition, it is important to note that in all of the frozen-time robustness results an increase in robustness requires a decrease in the allowable rate of variation and that the linear time-varying system fully inherits the robustness of the frozen-time family only as the allowable rate of variation becomes arbitrarily small.

The foregoing results relate to stability properties only. Were the linear dynamics, (2)-(3), an accurate approximation to the nonlinear dynamics, (1), then it might be expected that, when starting from the same initial conditions, the solutions of (1) and (2)-(3) remain correlated for some time. However, the solution to (2)-(3) is, in fact, only a *zeroeth order* approximation to the solution to (1) (see, for example, Leith & Leithead 1998b). This poor approximation property is inevitably reflected in the weakness of any approximation result; for example, Desoer & Wong (1968) and Desoer & Vidyasagar (1975 section 4.9). These results state that the peak absolute difference between the solution, $\delta\hat{\mathbf{x}}$, of the approximate system, (2)-(3), and the solution, $\delta\mathbf{x}$, of the nonlinear system, (1), is bounded provided the approximate system, (2)-(3), is stable, $\delta\mathbf{r}$ is sufficiently small and the initial conditions, $\delta\mathbf{x}(0)$ and $\delta\hat{\mathbf{x}}(0)$, are zero (although it is straightforward to extend this to encompass non-zero initial conditions which are sufficiently close to the origin). It can be seen that this result is, essentially, a restatement of local BIBO stability; that is, simply that the solutions of (1) and (2)-(3) both remain within a bounded region enclosing the origin provided the input and the initial conditions are sufficiently small.

2.2 Series expansion linearisation families

The foregoing results are confined to the dynamic behaviour locally to a *single* trajectory or equilibrium operating point. This is a significant limitation of the series expansion linearisation theory particularly since the local neighbourhood within which the analysis is valid may, in general, be very small. Within a gain-scheduling context, it is almost always required to consider the behaviour of a system relative to a *family* of operating points, which spans the envelope of operation, rather than relative to a single operating point. In order to increase the size of the operating region within which a series expansion linearisation is valid, it is therefore natural to consider combining, in some sense, the series expansion linearisations associated with a number of equilibrium points. At this point it is perhaps worth emphasising the clear distinction which exists between a single dynamic system and a family of dynamic systems, regardless of any superficial similarity between the two. The linear time-varying system (2)-(3),for example, is a quite different object (being a distinct dynamic system) from the associated family of frozen linear time-invariant systems (being a collection of dynamic systems). The importance of this distinction becomes particularly great when the state, input and/or output of the members of the family differ from one another as is the case when considering the family of series expansion linearisations of (1) relative to the equilibrium points. The state, input and output of each series expansion linearisation are perturbation quantities which depend on the equilibrium point considered. The relationship between the solution to a nonlinear system and the solutions to its series expansion linearisations is thus not straightforward when the system is not confined to the vicinity of a single equilibrium point. Nevertheless, it is possible to establish a weaker relationship. Namely, a relationship between the local stability of a nonlinear system and the stability of the associated series expansion equilibrium linearisations.

The relevant theory stems primarily from an early lemma by Hoppensteadt (1966) (see also Khalil & Kokotovic 1991, Khalil 1992 chapter 4, Lawrence & Rugh 1990) originally derived in the context of singular perturbation theory. Using this so-called frozen-input theory, it can be shown that the nonlinear system, (1), is locally BIBO stable in the vicinity of equilibrium operation provided that the members of its family of equilibrium linearisations are uniformly stable and the rate of variation is sufficiently slow. In addition, a trivial extension of this result is that, provided that the rate of variation is sufficiently slow, the nonlinear system also inherits the stability robustness of the equilibrium linearisations to smooth, finite dimensional, nonlinear perturbations (although there is the usual trade-off between robustness and the restrictiveness of the slow variation condition required). The slow variation condition in these results generally takes the form of a restriction both on the initial conditions of the system and on the rate of variation of the forcing input. This slow variation condition plays two roles: firstly, it ensures that the system stays sufficiently near to equilibrium operation (necessary owing to the use of equilibrium linearisations) and secondly, it ensures that the system evolves sufficiently slowly from the vicinity of one equilibrium point to the vicinity of another. It is emphasised that the analysis is *inherently* confined to a small neighbourhood enclosing the equilibrium operating points and consequently may be extremely conservative. Such a restriction is, of course, to be expected since the analysis is based on the properties of the series expansion linearisations of the nonlinear plant relative to the equilibrium points and so only utilises information regarding the dynamics at equilibrium. It is also worth noting that it is often difficult to test whether the stability conditions obtained are satisfied since they typically involve quantities which are difficult to evaluate (Shamma & Athans 1990). Indeed, perhaps owing to this difficulty, although frozen-input results are widely invoked in the literature to justify control designs it is quite rare for the theoretical slow variation conditions applying in a particular application to be actually determined.

An additional technical requirement in frozen-input stability analysis is that the equilibrium operating points are smoothly parameterised by the system input, $\mathbf{r}$. This requirement is not unduly restrictive in an analysis context but may be undesirable in the gain-scheduling design context, where it is more natural to parameterise the equilibrium operating points by the scheduling variable. It is perhaps interesting to note that Shamma & Athans (1990 section 4) apparently attempt to extend this type of analysis to a class of systems with a particular feedback structure and for which the family of equilibrium operating points is parameterised by a subset of the system states, $\mathbf{y}$, rather than the input, $\mathbf{r}$ (see Appendix C). The "output" $\mathbf{y}$ and the system dynamics mutually interact with one another whereas the input, $\mathbf{r}$, is independent of the system dynamics. Consequently, the analysis of an "output" scheduled nonlinear system is more difficult than the input-scheduled case. Shamma & Athans (1990) establish stability conditions requiring that $|\dot{\mathbf{y}}|$ is sufficiently small and, in addition, that the magnitude of the input, $\mathbf{r}$, and initial condition of a certain transformed state vector, ξ, are sufficiently small. The latter conditions confine the analysis, in general, to trajectories, (ξ(t), $\mathbf{r}$(t)) which lie within a small region enclosing the origin (see, for example, Shamma & Athans 1990, theorem 4.4). Since $\mathbf{y}$, which parameterises the equilibrium operating point, is a subset of the transformed states, ξ, (see Appendix C) it follows that $\mathbf{y}$(t) is also confined to a region enclosing the origin. Hence, the analysis cannot in general be applied to an extended family of equilibrium operating points as generally required in a gain-scheduling context.

2.3 Off-equilibrium linearisations

Conventional series expansion theory associates a linear time-invariant system only with equilibrium operating points. Consequently, any analysis/design based on this theory is generally only valid during near equilibrium operation. This limitation arises due to the characteristics of conventional series expansions but may be resolved by, instead, considering an alternative linearisation framework.

Before proceeding, in order to streamline the later discussion it is useful to explicitly highlight the linear and nonlinear dependencies of the dynamics by reformulating the nonlinear system (1) as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{r} + \mathbf{f}(\rho), \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{r} + \mathbf{g}(\rho) \tag{5}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, $\mathbf{D}$ are appropriately dimensioned constant matrices, $\mathbf{f}(\bullet)$ and $\mathbf{g}(\bullet)$ are nonlinear functions and $\rho(\mathbf{x},\mathbf{r})$ embodies the nonlinear dependence of the dynamics on the state and input with $\nabla_{\mathbf{x}}\rho$, $\nabla_{\mathbf{r}}\rho$ constant. Trivially, this reformulation can always be achieved by letting $\rho = [\mathbf{x}^T \;\; \mathbf{r}^T]^T$. However, the nonlinearity of the system is frequently dependent on only a subset of the states and inputs, in which case the dimension of ρ is reduced. The formulation, (5), defines a scheduling variable ρ which explicitly embodies the nonlinear dependence of the dynamics.

It may be shown (Leith & Leithead 1998b) that the solution to the velocity-based linearisation

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{6}$$

$$\dot{\hat{\mathbf{w}}} = (\mathbf{A}+\nabla\mathbf{f}(\rho_1))\,\hat{\mathbf{w}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1))\,\dot{\mathbf{r}} \tag{7}$$

$$\dot{\hat{\mathbf{y}}} = (\mathbf{C}+\nabla\mathbf{g}(\rho_1))\,\hat{\mathbf{w}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1))\,\dot{\mathbf{r}} \tag{8}$$

approximates the solution to the nonlinear system, (5) (and so (1)), locally to an operating point ($\mathbf{x}_1$, $\mathbf{r}_1$) at which $\rho_1=\rho(\mathbf{x}_1, \mathbf{r}_1)$. It is emphasised that ($\mathbf{x}_1$, $\mathbf{r}_1$) may be a general operating point (it need not be an equilibrium point and, indeed, may lie far from any equilibria). While the solution to the velocity-based linearisation is only a local approximation, there is a velocity-based linearisation associated with every operating point of a nonlinear system and the solutions to these linearisations may be pieced together to globally approximate the solution to the nonlinear system, (5), to an arbitrary degree of accuracy. Hence, the velocity-based linearisation family embodies the entire dynamics of a nonlinear system, with no loss of information, and is, in fact, an alternative representation of the nonlinear system. The velocity-based linearisation family is parameterised by the scheduling variable, ρ, and in this sense ρ captures the nonlinear structure of a system. The relationship between the nonlinear system, (5), and its velocity-based linearisation, (6)-(8), is direct. Differentiating (5), an alternative representation of the nonlinear system is

$$\dot{\mathbf{x}} = \mathbf{w} \tag{9}$$

$$\dot{\mathbf{w}} = (\mathbf{A}+\nabla\mathbf{f}(\rho))\mathbf{w} + (\mathbf{B}+\nabla\mathbf{f}(\rho))\,\dot{\mathbf{r}} \tag{10}$$

$$\dot{\mathbf{y}} = (\mathbf{C}+\nabla\mathbf{g}(\rho))\mathbf{w} + (\mathbf{D}+\nabla\mathbf{g}(\rho))\,\dot{\mathbf{r}} \tag{11}$$

Evidently, the velocity-based linearisation, (6)-(8), is simply the frozen form of (9)-(11) at the operating point, ($\mathbf{x}_1$, $\mathbf{r}_1$).

The relationship between the solution to a nonlinear system and the solutions to the members of the associated velocity-based linearisation family can be used to derive conditions relating the stability of a nonlinear system to the stability of its velocity-based linearisations. General stability analysis methods such as small gain theory and Lyapunov theory can be applied to derive velocity-based stability conditions (including the methods in section 4). In addition, by adopting the velocity-based framework, it is possible to extend and strengthen the classical frozen-input stability results discussed in section 2.2. Specifically, BIBO stability of the nonlinear system (1) is guaranteed provided the members of its velocity-based linearisation family are uniformly stable, unboundedness of the state **x** implies that **w** is unbounded (assuming the input **r** is bounded) and the class of inputs and initial conditions is restricted to limit the rate of evolution of the nonlinear system to be sufficiently slow (Leith & Leithead 1998b). In addition, provided that the rate of evolution is sufficiently slow, the nonlinear system inherits the stability robustness of the members of the velocity-based linearisation family (Leith & Leithead 1998b) (with the usual trade-off between robustness and the restrictiveness of the slow variation condition required).. This velocity-based result involves no restriction to near equilibrium operation other than that implicit in the slow variation requirement; for example, for some systems where the slow variation condition is automatically satisfied the class of allowable inputs and initial conditions is unrestricted and the stability analysis is global.

3. Divide & Conquer Gain-Scheduling Design

Gain-scheduling design approaches conventionally construct a nonlinear controller, with certain required dynamic properties, by combining, in some sense, the members of an appropriate family of linear time-invariant controllers. Design approaches may be broadly classified according to the linear family utilised. Classical gain-scheduling design approaches, based on the series expansion linearisation of a system about its equilibrium points, are discussed in section 3.1. Recent, and closely related, approaches based on neural/fuzzy modelling and off-equilibrium linearisations are considered, respectively, in sections 3.2 and 3.3.

3.1 Classical gain-scheduling design

Consider the nonlinear plant with dynamics, (1). The classical gain-scheduling design approach is based on the family of equilibrium linearisations of the plant and may be applied directly to a broad range of nonlinear systems. The design procedure typically involves the following steps (see, for example, Astrom & Wittenmark 1989 section 9.5, Shamma & Athans 1990, Hyde & Glover 1993).

1. The equilibrium operating points of the plant are parameterised by an appropriate quantity, ρ, which may involve the plant input, output and/or state.

2. The plant dynamics, (1), are approximated, locally to a specific equilibrium operating point, $(\mathbf{x}_o,\mathbf{r}_o,\mathbf{y}_o)$, at which ρ equals ρ_o, by the series expansion linearisation,

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))\delta\hat{\mathbf{x}} + \nabla_r\mathbf{F}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))\delta\mathbf{r} \qquad (12)$$

$$\delta\hat{\mathbf{y}} = \nabla_x\mathbf{G}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))\delta\hat{\mathbf{x}} + \nabla_r\mathbf{G}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))\delta\mathbf{r} \qquad (13)$$

$$\delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}(\rho_o),\ \hat{\mathbf{y}} = \delta\hat{\mathbf{y}} + \mathbf{y_o}(\rho_o),\ \delta\hat{\mathbf{x}} = \hat{\mathbf{x}} - \mathbf{x}_o(\rho_o) \qquad (14)$$

3. For a suitable controller input, **e**, with equilibrium value, $\mathbf{e_o}(\boldsymbol{\rho_o})$, a linear time-invariant controller is designed,

$$\delta\dot{\mathbf{z}} = \mathbf{A}(\boldsymbol{\rho_o})\delta\mathbf{z} + \mathbf{B}(\rho_o)\delta\mathbf{e} \qquad (15)$$

$$\delta\mathbf{r} = \mathbf{C}(\rho_o)\delta\mathbf{z} + \mathbf{D}(\rho_o)\delta\mathbf{e} \qquad (16)$$

$$\delta\mathbf{e} = \mathbf{e} - \mathbf{e_o}(\rho_o),\ \mathbf{r} = \delta\mathbf{r} + \mathbf{r_o}(\rho_o) \qquad (17)$$

   which ensures appropriate closed-loop performance when employed with the plant linearisation, (12)-(14). It should be noted that ρ_o is assumed to be constant when designing this linear controller. (While the controller output equation, (16), is formulated assuming that every input to the plant is a controller output, it is, of course, straightforward to accommodate situations where the controller output is only a subset of the inputs; for example, when the plant is subject to external disturbances).

4. Repeat steps 2 and 3 as required for a family of equilibrium operating points. A family of linear time-invariant controllers is obtained corresponding to the family of equilibrium operating points; both the controller family and the equilibrium operating points are parameterised by ρ. When a smoothly gain-scheduled controller is required, the linear controller designs are selected to have compatible structures which permit smooth interpolation between the designs.

   With regard to the theoretical basis for smooth interpolation (as opposed to non-smooth), Kamen *et al.* (1989) note that when the plant linearisations are stabilizable at every equilibrium point, there exists a stabilising family of static state feedback controllers the gains of which are continuously differentiable functions of the plant matrices, $\nabla_x\mathbf{F}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))$, $\nabla_r\mathbf{F}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))$, $\nabla_x\mathbf{G}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))$, $\nabla_r\mathbf{G}(\mathbf{x_o}(\rho_o), \mathbf{r_o}(\rho_o))$. More generally, when the plant linearisations are controllable at every equilibrium point, the poles of the members of the closed-loop family can always be assigned by utilising a continuous family of dynamic state feedback controllers (Sontag 1985).

With regard to constructive interpolation procedures, Stilwell & Rugh (1998) propose an interpolation approach for state feedback and state feedback/observer based controllers which explicitly ensures stability of the closed-loop system at the interpolated operating points (at the cost of some conservativeness). However, relatively *ad hoc* interpolation schemes appear to be more commonly employed in practice. In particular, while more general approaches are certainly possible (see, for example, the fuzzy interpolation schemes proposed by Zhao *et al.* 1993, Qin & Borders 1994), it is usual to simply linearly interpolate either the elements of the matrices in the state-space representation of the controller (*e.g.* Hyde & Glover 1993) or the gain, poles and zeroes of the controller transfer functions (*e.g.* Nichols *et al.* 1993). The number of members in the linear controller family employed is increased until satisfactory characteristics are obtained at the interpolated points. It should be noted that there seem to be few theoretical results relating to the use of linear interpolation. (With the notable exception of Shahruz & Behtash (1992) in the specific context of full-state feedback pole-placement control for a class of linear time-varying systems (namely LPV systems, see section 6)).

5. Implement a nonlinear controller based on the family of linear controllers, (15)-(17). This step is of considerable importance since the choice of nonlinear controller realisation can greatly influence the closed-loop performance (Leith & Leithead (1996), for example, observe in the context of wind turbine regulation that the performance improvement obtained by employing a nonlinear gain-scheduled controller can be entirely lost by adopting a poor choice of nonlinear controller realisation). There are three main approaches to realising a nonlinear gain-scheduled controller[2].

   a. *Classical approach*

   Firstly, implement the controller input and output transformations, (17). It should be noted that this step is often implicit. Typically, the controller input, $\mathbf{e}=\mathbf{y}-\mathbf{y}_{ref}$, is zero (or near zero) in equilibrium (that is, $\mathbf{e_o}(\boldsymbol{\rho_o})=0$ and $\delta\mathbf{e} = \mathbf{e}$) and either the plant exhibits pure integral action, so that $\mathbf{r_o}(\rho)$ is identically zero, or each linear controller contains integral action which implicitly generates $\mathbf{r_o}(\rho)$ through the action of the feedback loop (see, for example, Astrom & Wittenmark 1989 section 9.5, Shamma & Athans 1990, Hyde & Glover 1993). Alternatively, the controller output transformation may be implemented by explicitly calculating the equilibrium controller output as a function of ρ (see, for example, Rugh 1991, Shamma & Athans 1990). However, the latter approach may involve rather complex calculations which are sensitive to modelling errors and, consequently, seems to be largely of theoretical interest (Hyde & Glover 1991).

   Secondly, substitute ρ (or some related quantity) for ρ_o in the family of local linear controllers, (15)-(17), to obtain a nonlinear controller. It is noted that the scheduling variable need not be continuous; for example, it may be piece-wise constant, corresponding to switching between the members of the family of local linear controllers. Typically, the selection of an appropriate scheduling variable is based on physical insight (Astrom & Wittenmark 1989).

   b. *Local linear equivalence*

   A nonlinear controller realisation is selected for which the series expansion linearisation at each equilibrium operating point matches the relevant member, (15)-(17), of the linear controller family (and of course such that the family of controller equilibrium inputs and outputs agree with $\mathbf{e_o}(\rho)$ and $\mathbf{r_o}(\rho)$). It should be noted that nonlinear controller realisations obtained by traditional approaches (see preceding paragraph) need *not* satisfy this local linear equivalence condition. Wang & Rugh (1987a,b) give necessary and sufficient existence conditions for a linear controller family to be a linearisation family; that is, to correspond to the equilibrium linearisations of some nonlinear controller. These are extended to infinite dimensional systems by Banach & Baumann (1990). On the face of it, the existence conditions appear to be rather restrictive. However, it is very common for controllers to contain integral action to ensure rejection of steady disturbances and in this case the existence conditions are readily satisfied (Kaminer *et al.* 1995, Lawrence & Rugh 1995). Specific classes of controller without integral action are also considered by Rugh (1991), Lawrence & Rugh (1995) but the proposed nonlinear controller realisations require *a priori* knowledge of the equilibrium controller output as a function of ρ in order to explicitly implement the input and output transformations, (17). Hence, they may involve rather complex calculations which are sensitive to modelling errors (Hyde & Glover 1991).

[2] The classical gain-scheduling approach is based on the plant linearisations at a number of equilibrium points and so utilises only information regarding the plant dynamics at equilibrium. The discussion in this section is therefore confined to controller realisation approaches based only on knowledge of the plant dynamics near equilibrium. Methods which exploit additional information, particularly regarding the plant dynamics at non-equilibrium operating points, may lead to controller realisations which achieve better performance but are outwith the scope of the conventional gain-scheduling approach; see later sections.

Given that the classical gain-scheduling approach only employs information about the plant dynamics locally to the equilibrium points, the local linear equivalence condition seems to be quite reasonable and an improvement over traditional nonlinear controller realisation approaches. However, the local linear equivalence condition only requires that the nonlinear controller realisation has the required equilibrium linearisations. The neighbourhoods within which the linearisations are an accurate description of the controller dynamics is not addressed. There exists an infinite number of nonlinear controller realisations satisfying the local linear equivalence condition and the neighbourhoods within which the linearisations are valid may be relatively large for some realisations but vanishingly small for others. The local linear equivalence condition does not distinguish between these realisations. As discussed in section 2.2.2, since the controller is designed on the basis only of the plant dynamics locally to the equilibrium points, a restriction to near equilibrium operation is inherent to the classical gain-scheduling approach. However, a further restriction may be required to ensure that the system remains sufficiently near to equilibrium that the dynamics of the nonlinear controller are similar to those of the controller equilibrium linearisations; that is, to the linear controller family designed at step 4. This latter restriction could be much stronger than that associated with the plant. The local linear equivalence condition entirely ignores this restriction: it may be very strong for some controller realisations satisfying the local linear equivalence condition but relatively weak for others. The local linear equivalence condition provides no guidance for distinguishing between these realisations and this greatly diminishes its utility (Leith & Leithead 1998d, 1999c).

c. *Extended local linear equivalence*

In order to address the deficiencies in the local linear equivalence approach, Leith & Leithead (1996, 1998a) propose that the nonlinear controller realisation should be selected to satisfy an extended local linear equivalence condition. The extended local linear equivalence condition selects, from the class of nonlinear controller realisations satisfying the local linear equivalence condition, those controller realisations which maximise, in an appropriate sense, the neighbourhoods within which the controller equilibrium linearisations are an accurate description of the controller dynamics. Specifically, controller realisations are selected for which the individual neighbourhoods are unbounded and their union covers the entire operating space, Φ. Unnecessary restrictions on the operating envelope of the closed-loop system are thereby relaxed (in marked contrast to realisations satisfying the local linear equivalence condition only). Whilst the extended local linear equivalence condition is a strong requirement, controller realisations satisfying this condition for a very wide class of MIMO linear controller families are derived in Leith & Leithead (1998a). Similarly to the local linear equivalence condition, integral action has an important role in facilitating the realisation of nonlinear controllers satisfying the extended local linear equivalence condition.

The classical gain-scheduling approach only employs information about the plant dynamics locally to the equilibrium points. In these circumstances, the extended local linear equivalence condition provides useful guidance to selecting an appropriate nonlinear controller realisation and seems in some sense to be the "best" that can be achieved within the limitations imposed by knowing only equilibrium information (Leith & Leithead 1998a). However, this certainly does not mean that other choices of controller realisation may not be superior, only that the analytic selection of such realisations seems to require further information about the plant dynamics. Methods for using additional information, particularly regarding the plant dynamics at non-equilibrium operating points, are outwith the scope of the classical gain-scheduling approach and are discussed in later sections.

The foregoing gain-scheduling design process is frequently iterative, with the controller revised in the light of subsequent analysis until a satisfactory design is achieved.

The effectiveness of the classical gain-scheduling design approach depends on the dynamic characteristics of the nonlinear system, composed of the nonlinear plant and the nonlinear gain-scheduled controller, being related to those of the members of an associated family of linear systems, composed of the plant linearisations and corresponding local linear controllers. Traditionally, the results reviewed in sections 2.1-2.2 are applied to relate the dynamic characteristics of a nonlinear system to those of such an associated family of linear systems. Of course, both series expansion linearisation theory (section 2.1.1) and frozen-input theory (section 2.2.2) are inherently restricted to operation near equilibrium. With regard to the support provided for the gain-scheduling design procedure, it is noted that these results consider only the stability properties of the nonlinear system and provide little direct insight into other dynamic properties, such as the transient response. In addition, although frozen-input theory can support the stability analysis of a nonlinear gain-scheduled control system about a family of equilibrium points, it provides little insight into the controller design procedure since the *frozen-input* representation of the controlled system is quite distinct from the mixed series-expansion/frozen-scheduling variable representation employed in step 3 of the design procedure. *Series expansion* linearisations of the plant are employed but the corresponding local controller designs are *frozen-scheduling variable* linearisations of the resulting nonlinear controller. In contrast, when analysing the dynamic behaviour of the controller locally to a single equilibrium operating point using series expansion linearisation theory, the *series expansion* linearisation is employed instead of the frozen-scheduling variable linearisation. The frozen-input analysis of the controlled system in the vicinity of the family of equilibrium operating points does not reduce to either the series expansion analysis (see, for example Shamma 1988 p110) or the mixed series-expansion/frozen-scheduling variable analysis employed in the design procedure. The analysis of gain-scheduling design by means of conventional results, relating the dynamic characteristics of a nonlinear system to those of an associated family of linear systems, is, therefore, rather convoluted and inefficient. The use of a variety of different local approximations obscures insight and is unnecessary.

Remark *Control of linear time-varying systems*

Gain-scheduled controllers are designed classically on the basis of the plant linearisations at a number of equilibrium operating points. Of course, it follows from series expansion linearisation theory that a nonlinear system can be approximated by a linear system in the vicinity of a trajectory as well as in the vicinity of an equilibrium point. The linearisation relative to a trajectory is a linear time-varying, rather than linear time-invariant, system. Although rather less well developed than the control design methods for linear time-invariant systems, control design methods for linear time-varying systems are considered in, for example, Ravi *et al.* (1991), Voulgaris & Dahleh (1995), Scherpen & Verhaegen (1996) (see also the methods discussed in section 4 below). Whilst these approaches typically take a linear time-varying plant as the starting point for control design, Sontag (1987a,b) specifically considers control design for the family of linear time-varying systems corresponding to the series expansion linearisations of a nonlinear plant relative to a family of trajectories for control design. However, a fundamental limitation of any such approach for nonlinear systems is that the controller obtained is only valid in some small neighbourhood about a specific reference trajectory. In the classical gain-scheduling approach, an attempt is made to enlarge the allowable operating region by combining the linear controllers associated with a number of the plant equilibrium linearisations. However, there do not appear to be comparable results in the literature relating to the combination of the linear time-varying controllers associated with the plant linearisations about a number of trajectories with the aim of enlarging the allowable operating region.

3.2 Neural/fuzzy gain-scheduling

Despite the widespread application of gain-scheduled controllers, it is evident from the preceding section that the classical techniques for the design of gain-scheduled systems remain rather poorly developed. In particular, a limitation of these techniques is that they exploit the behaviour only in the vicinity of the equilibrium operating points (this generally imposes an inherent slow variation requirement on the system to ensure that the state remain close to equilibrium which is additional to any slow variation requirement associated with the change in linearised dynamics as the system moves from the vicinity of one equilibrium point to another). However, in order to meet increasingly stringent performance objectives, gain-scheduled controllers are frequently required to operate both during transitions between equilibrium operating points (which might transiently take the system far from equilibrium) and during sustained operation far from equilibrium. A number of approaches developed in the fuzzy-logic and neural network literatures attempt to relax restrictions to near equilibrium operation while remaining closely related to the classical gain-scheduling design philosophy. These design approaches typically involve the following steps.

1. The plant dynamics are formulated as a blended multiple model representation such as a Takagi-Sugeno model or local model network; that is, in the continuous-time case, a system of the form

$$\dot{\mathbf{x}} = \sum_i \overline{\mathbf{F}}_i(\mathbf{x},\mathbf{r})\,\mu_i(\rho)\,,\ \ \mathbf{y} = \sum_i \overline{\mathbf{G}}_i(\mathbf{x},\mathbf{r})\,\mu_i(\rho) \tag{18}$$

   Systems of this form are considered by, for example, Gawthrop (1995) and Shorten *et al.* (1999) and are closely related to the systems considered by Johansen & Foss (1993), Johansen (1994), Tanaka & Sano (1994), Wang *et al.* (1996), Hunt & Johansen 1997, Tanaka *et al.* (1998), Kiriakidis (1999), Slupphaug & Foss (1999). The local models associated with the blended multiple model system,

$$\dot{\mathbf{x}}_i = \overline{\mathbf{F}}_i(\mathbf{x}_i,\mathbf{r})\,,\ \ \mathbf{y}_i = \overline{\mathbf{G}}_i(\mathbf{x}_i,\mathbf{r}) \tag{19}$$

   are blended via the scalar weighting (or validity) functions μ_i. The latter are typically normalised such that $\sum_i \mu_i(\rho) = 1$ with the quantity $\rho(\mathbf{x},\mathbf{r})$ embodying the dependence of the blending on the state and input.
2. The blended representation, (18), immediately suggests a natural divide and conquer type of control design approach whereby a local controller is designed for each local model, (19).
3. The local controller designs are then blended, using the weighting functions μ_i, to obtain a nonlinear controller with similar form to the plant, (18).

In order to maintain the continuity with linear methods which is a primary motivation of gain-scheduling design approaches, it is attractive to consider plant and controller representations with linear local models (see, for example, Gawthrop 1995, Tanaka & Sano 1994, Wang *et al.* 1996, Tanaka *et al.* 1998, Kiriakidis 1999). Blended representations of the form, (18), are known to be universal approximators (see, for example, Johansen & Foss 1993); that is, any smooth nonlinear system, (1), may be reformulated as in (18). However, available approximation results, which are largely based on series expansion theory, are confined to situations where the local models are inhomogeneous. For example, it is common to consider affine local models

$$\overline{\mathbf{F}}_i(\mathbf{x},\mathbf{r}) = \overline{\alpha}_i + \overline{\mathbf{A}}_i\mathbf{x} + \overline{\mathbf{B}}_i\mathbf{r}\,,\ \ \overline{\mathbf{G}}_i(\mathbf{x},\mathbf{r}) = \overline{\beta}_i + \overline{\mathbf{C}}_i\mathbf{x} + \overline{\mathbf{D}}_i\mathbf{r} \tag{20}$$

here $\overline{\alpha}_i, \overline{\mathbf{A}}_i, \overline{\mathbf{B}}_i, \overline{\beta}_i, \overline{\mathbf{C}}_i, \overline{\mathbf{D}}_i$ are constants. It is emphasised that, owing to the inhomogeneous terms $\overline{\alpha}_i, \overline{\beta}_i$, local models of the form (20) do not exhibit the superposition property and are thus *not* linear. Moreover, it is not generally admissible to simply neglect the inhomogeneous terms for control design purposes, even when the controller contains integral action (it is not difficult, for example, to construct examples where a blended controller designed in such a manner leads to an unstable closed-loop system). The consideration of design methods for affine and other types of nonlinear controller is outwith the scope of the present paper and is not pursued further here (see, for example, Hunt & Johansen 1997, Slupphaug & Foss 1999). Although standard approximation results do not support the general use of linear local models, it is nevertheless clear that blended representations with linear local models can approximate the particular class of nonlinear systems with the (quasi-LPV) form

$$\dot{\mathbf{x}} = \mathbf{A}(\rho)\mathbf{x} + \mathbf{B}(\rho)\mathbf{r},\ \ \mathbf{y} = \mathbf{C}(\rho)\mathbf{x} + \mathbf{D}(\rho)\mathbf{r} \tag{21}$$

such as those considered by Tanaka & Sano (1994), Wang *et al.* (1996), Tanaka *et al.* (1998), Kiriakidis (1999). The class of nonlinear systems, (1), which may be reformulated in this form is considered further in section 4.

Design methods based on Takagi-Sugeno and local model network representations are certainly not confined to divide and conquer type approaches (LMI-based LPV methods similar to those discussed in section 4 are applied by, for example, Tanaka & Sano 1994, Wang *et al.* 1996, Tanaka *et al.* 1998, Kiriakidis 1999). The blended structure of the representations is, nevertheless, naturally associated with divide and conquer methods. These representations are sometimes employed in situations where the operating space of a nonlinear system can be decomposed into a number of distinct operating regions within which the dynamics are described by a particular model. Each model in the blended multiple model system is valid in an extended operating region and blending is confined to small transition regions at the boundaries between these main operating regions. In this context, blending plays a relatively minor role and the approach is more akin to piecewise approximation. However, perhaps a more flexible approach is to let each local model describe the dynamics of a nonlinear system in some small region of the operating space. The role of the blending is then to provide smooth interpolation, in some sense, between the local models with the aim of achieving an accurate representation with only a small number of local models. The blending is, therefore, central to the utility of the approach. Since the operating regions, in which no single local model dominates and several local models contribute significantly to the blended multiple model system, constitutes the greater part of the operating space, the characteristics of the blended multiple model system in these regions is of primary concern. In order to facilitate analysis and design, it is attractive for the dynamics of the blended multiple model system in these regions to be directly related to the local models (this property, for example, clearly underlies steps 2 and 3 in the gain-scheduling design procedure outlined earlier in this section). While the algebraic relationship between the right-hand sides of the nonlinear blended system, (18), and the local models, (19), is direct, the relationship between the solutions to these dynamic systems is less clear. It follows immediately from the approximation theory discussed in section 2 that the solution to the nonlinear blended system is approximated, locally to any operating point, by its first-order series expansion. It is easily seen that the first-order expansion of (18) will contain cross-terms involving the derivatives of the weighting functions, μ_i,. Hence, the relationship between the dynamics of the first-order series expansions (and so the nonlinear blended system) and those of the local models is generally not straightforward. In particular, the dynamics of the blended multiple model system are *not* described by a simple blend of dynamics of the local models. Indeed, in operating regions where the derivative, $\nabla\mu_i$, of any validity function is large, the dynamics of the first-order series expansions may be strongly influenced by the varying nature of the validity functions and only weakly related to the dynamics of the local models (Shorten *et al.* 1999, Leith & Leithead 1999a). The applicability of divide and conquer design approaches based on Takagi-Sugeno/local model network representations appears, consequently, to be restricted to situations where either blending is minimal or the evolution of the system is sufficiently slow that the influence of the cross-terms involving $\nabla\mu_i$ may be neglected. It should be noted that the latter slow variation condition is quite distinct from, and may be additional to, the slow variation conditions associated with classical gain-scheduling approaches.

3.3 Gain-scheduling using off-equilibrium linearisations

The restriction to near equilibrium operation in classical gain-scheduling approaches essentially arises as a result of the limitations of conventional series expansion linearisation theory. Specifically, conventional series expansion linearisation theory associates a linear system only with equilibrium operating points. This limitation is directly addressed by the velocity-based linearisation approach (section 2.3) which generalises series expansion linearisation to associate a linear system with every operating point of a nonlinear system, not just the equilibrium operating points. The velocity-based linearisation of the cascade combination of two nonlinear systems is simply the cascade combination of the corresponding velocity-based linearisations of the two nonlinear systems (Leith & Leithead 1998c). Similarly for the parallel and closed-loop combinations of two nonlinear systems. These properties immediately suggests the following generalised gain-scheduling design procedure.

1. Determine the members

$$\dot{\hat{\mathbf{x}}} = \hat{\mathbf{w}} \tag{22}$$

$$\dot{\hat{\mathbf{w}}} = (\mathbf{A}+\nabla\mathbf{f}(\rho_1)\,\nabla_{\mathbf{x}}\rho\,)\,\hat{\mathbf{w}} + (\mathbf{B}+\nabla\mathbf{f}(\rho_1)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{23}$$

$$\dot{\hat{\mathbf{y}}} = (\mathbf{C}+\nabla\mathbf{g}(\rho_1)\,\nabla_{\mathbf{x}}\rho\,)\,\hat{\mathbf{w}} + (\mathbf{D}+\nabla\mathbf{g}(\rho_1)\,\nabla_{\mathbf{r}}\rho\,)\,\dot{\mathbf{r}} \tag{24}$$

   of the velocity-based linearisation family associated with the nonlinear plant dynamics, (5).

2. Based on the velocity-based linearisation family of the plant, determine the required velocity-based linearisation family of the controller such that the resulting closed-loop family achieves the performance requirements. Since each member of the plant family is linear, conventional linear design methods can be utilised to design each corresponding member of the controller family. Alternatively, the LPV design methods discussed in section 4 might be employed.

   Since a velocity-based linearisation is associated with each value of ρ, the velocity-based linearisation family has an infinite number of members. It is, of course, attractive to consider design procedures which are based on only a small finite number of members of the plant velocity-based linearisation family and a similar issue arises in conventional gain-scheduling approaches since there are generally a continuum of equilibrium operating points and so an infinite number of equilibrium linearisations. In conventional gain-scheduling approaches, this is resolved by interpolating between a small number of linear controller designs and a similar approach can be employed in the velocity-based gain-scheduling design procedure. Unfortunately, the interpolation approaches utilised in conventional gain-scheduling are often rather arbitrary (*e.g.* linear interpolation of poles/zeroes or state-space matrices). An alternative is to consider approximating the velocity-based linearisation family of the plant by interpolating between a small finite number of velocity-based linearisations (denoted the "local models"). The local models and interpolating functions are selected to ensure that the approximate family is a sufficiently accurate approximation. A linear controller may then be designed for each local model and the controller velocity-based linearisation family obtained by interpolating between the linear controllers. By using the same interpolating functions as in the approximate plant, the interpolated controller family reflects the plant characteristics. Moreover, in contrast to the situation with Takagi-Sugeno/local model network blended representations, a direct relationship is maintained between the dynamic characteristics of the interpolated velocity-based linearisations and the dynamic characteristics of the local models thereby facilitating analysis and design (Leith & Leithead 1999a).

3. Realise a nonlinear controller with the velocity-based linearisation family designed at step 2 (Leith & Leithead 1998c, 1999a,b). The velocity-based form of the controller can be obtained directly from the family of linear controllers by simply permitting the scheduling variable, ρ, in the velocity-based linearisation family to vary with the operating point. Of course, it is necessary to address a number of practical issues relating to this velocity-form. Specifically, some care is required when realising the velocity-form since the input is $\dot{\mathbf{r}}$ rather than $\mathbf{r}$. Moreover, owing to the differentiation and integration operations associated with the velocity-form local models, the order of the velocity-form may be greater than that of the direct representation, (5).

This design procedure retains a divide and conquer approach and maintains the continuity with linear design methods. In addition, similarly to classical gain-scheduling approaches, the velocity-based gain-scheduling approach provides an open framework in the sense that there is no inherent restriction to any particular linear design methodology. However, in contrast to classical gain-scheduling approaches, the velocity-based approach employs a streamlined analysis and design framework which utilises a single type of linearisation throughout thereby facilitating analysis and design. Moreover, the resulting nonlinear controller may be valid throughout the operating envelope of the plant, not just in the vicinity of the equilibrium operating points (see, for example, Leith & Leithead 1999b). This is a direct consequence of employing the velocity-based linearisation framework rather than the conventional series expansion linearisation about an equilibrium operating point.

The stability and performance of nonlinear systems designed by the velocity-based gain-scheduling approach can be analysed using any of the available general-purpose methods for nonlinear systems such as small gain theory and Lyapunov theory (also frozen-scheduling variable theory as discussed in section 2.3). Of course, the results obtained are generally conservative since, except in special cases, only sufficient conditions are known for the stability of nonlinear systems. However, since the design approach is not restricted to any particular analysis method, the designer is free to employ those methods which are most suited to a particular application.

Remark For the class of nonlinear plants satisfying the extended local linear equivalence condition (see section 3.1), the dynamics are embodied by the equilibrium linearisations together with knowledge of the scheduling variable, ρ. Under these circumstances, the classical and velocity-based divide and conquer design methods are equivalent provided an appropriate controller realisation is adopted (Leith & Leithead 1998c).

4. LPV gain-scheduling

The gain-scheduling approaches discussed in the previous section are linked by a divide and conquer design philosophy, whereby the design of a nonlinear controller is decomposed into the design of a number of linear controllers. Continuity is thus maintained with well-established linear methods for which there is considerable theory and accumulated practical experience. In addition, it should be noted that these approaches provide open design frameworks in the sense that there is no inherent restriction to any particular linear design methodology. In recent years, a number of interesting alternative approaches have been proposed in the context of gain-scheduling design (for example, Becker *et al.* 1993, Packard 1994, Apkarian & Adams 1998). Since these approaches share the use of linear parameter-varying (LPV) plant representations (rather the nonlinear representation, (1)), they are commonly referred to as LPV gain-scheduling methods. It should be noted that most of these approaches are conceptually quite distinct from the conventional gain-scheduling approach since they involve the direct synthesis of a controller rather than its construction from a family of local linear controllers designed by linear time-invariant methods. Moreover, rather than providing open frameworks, the LPV approaches typically utilise norm based performance measures. In particular, the induced L_2 norm is widely employed as a performance measure since this enables a degree of continuity to be maintained with linear H_∞ theory in the sense that when the plant is linear time-invariant the approaches are equivalent to linear H_∞ design. However, all of the LPV approaches at present involve some degree of conservativeness and correspond to special extensions, rather than genuine generalisations, of linear H_∞ design.

This section is organised as follows. Firstly, the results relating to the representation of nonlinear systems in LPV form are discussed in section 4.1. The LPV gain-scheduling procedures are then reviewed. These approaches can be categorised according to whether they utilise a small-gain approach or a Lyapunov-based approach: the former are reviewed in section 4.2 and the latter in section 4.3.

4.1 LPV systems

Perhaps the earliest work on linear parameter-varying (LPV) systems in the context of gain-scheduling is that of Shamma (1988), Shamma & Athans (1991). Shamma (1988), Shamma & Athans (1991) consider systems

$$\dot{\mathbf{x}} = \mathbf{A}(\theta(t))\mathbf{x}+\mathbf{B}(\theta(t))\mathbf{r}, \quad \mathbf{y}=\mathbf{C}(\theta(t))\mathbf{x}+\mathbf{D}(\theta(t))\mathbf{r} \tag{25}$$

where the parameter, θ, is an exogenous time-varying quantity (strictly independent of the state $\mathbf{x}$ of the system) which takes values in some allowable set. Under these conditions, an LPV system is simply a particular form of linear time-varying system. However, whilst a number of subsequent gain-scheduling approaches have taken LPV systems as their starting point, it is *a priori* far from clear that this class of systems is sufficiently rich to include a reasonable range of practical gain-scheduling applications. Linear time-varying representations of nonlinear systems are largely associated with series expansion linearisation theory, reviewed in section 2.1. Consider the nonlinear system, (1). It follows from series expansion linearisation theory that the nonlinear system may be approximated by the linear time-varying system, (2)-(52). However, for this approximation to be accurate there is a requirement that $|\delta\mathbf{x}|$ and $|\delta\mathbf{r}|$ are sufficiently small; that is, the series expansion linearisation is only valid within a small neighbourhood about the specific equilibrium operating point or trajectory ($\tilde{\mathbf{x}}(t)$, $\tilde{\mathbf{r}}(t)$, $\tilde{\mathbf{y}}(t)$).

In order to increase the size of the operating region within which a series expansion linearisation based LPV representation is valid, it is perhaps attractive to consider combining, in some sense, the series expansion linearisations associated with a number of equilibrium points/trajectories. However, it is important to make a clear distinction between the LPV system, (25), and the family of linear systems associated with the equilibrium points/trajectories of a nonlinear system, (1). Clearly, the linearisation family is a collection of dynamic systems whilst the LPV system is a single dynamic system. The state, input and output of a series expansion linearisation are perturbation quantities which depend on the equilibrium point/trajectory considered. Hence, the members of the linearisation family each have different state, input and output in general and *cannot* be directly combined to obtain an LPV system; see the discussion in section 2.2.1. (Nevertheless, in order to obtain a representation which is in LPV form, the varying state, input and output transformations appear to be neglected, for example, in the missile and aircraft models employed for LPV gain-scheduling design by, for example, Apkarian *et al.* 1995, Spillman *et al.* 1996, Fialho *et al.* 1997, Lee & Spillman 1997. While satisfactory results appear to be achieved in the specific examples considered by these authors, it is not difficult to construct other examples where a control design based on such an *ad hoc* formulation leads to an unstable closed-loop system when applied to the original plant).

Conventional series expansion theory does not support the reformulation of general nonlinear systems in LPV form without strong restrictions on the operating region. However, following a similar approach to Helmersson (1995b chapter 10) (see also Scherer *et al.* 1997, Apkarian & Adams 1998 section IV), consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x},\mathbf{r})\mathbf{x}+\mathbf{B}(\mathbf{x},\mathbf{r})\mathbf{r}, \quad \mathbf{y}=\mathbf{C}(\mathbf{x},\mathbf{r})\mathbf{x}+\mathbf{D}(\mathbf{x},\mathbf{r})\mathbf{r} \tag{26}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$ and the input and initial conditions of the state are restricted such that the solution $\mathbf{x}(t)$ is confined to some operating region $X \subset \Re^n$. It is immediately evident that the solutions to the nonlinear system, (26), are a subset of the solutions to the LPV system, (25), with $\theta \in X$. (Since the parameter θ can vary arbitrarily in (25), the solutions to (26) are just the solutions to (25) associated with particular parameter trajectories). Hence, whilst it is generally not possible to reformulate a nonlinear system as an LPV system, it is possible to over-bound the nonlinear system, (26), by an LPV system in the sense that every solution to the nonlinear system is a solution to the LPV system (but not *vice versa*). Of course some degree of conservativeness can be expected with such an approach. In addition, it is emphasised that support for arbitrary parameter variations in the LPV system is an essential feature of the over-bounding; for example, it is not possible to arbitrarily restrict the rate of variation of the parameter, $\dot{\theta}$. Generalising the definition of Shamma (1988), nonlinear systems, (26), are referred to here as quasi-LPV systems. Of course, it still remains to be established whether this class is sufficiently rich to include a reasonable range of gain-scheduling applications.

Various approaches proposed for reformulating nonlinear systems into quasi-LPV form are reviewed in Appendix C. In summary, these theoretical results do *not* support the representation of nonlinear dynamic systems in quasi-LPV form without, in general, considerable restrictions on the class of nonlinear systems considered or the allowable operating region. However, the restrictions on the operating region essentially arise from the limitations of series expansion linearisation theory. Consider, instead, the velocity-based linearisation framework in Leith & Leithead (1998a) reviewed in section 2.3. The velocity-based linearisation is a generalisation of the conventional series expansion linearisation. Since there is a velocity-based linearisation associated with every operating point of a nonlinear system, rather than just the equilibrium operating points, the velocity-based framework resolves one of the primary limitations of conventional series expansion linearisation theory. In the present context, it is noted that the velocity representation, (9)-(11), is a particular type of quasi-LPV system. It follows immediately that every nonlinear system, (1), can be reformulated as an equivalent quasi-LPV system. The reformulation is valid for a very general class of nonlinear systems and it is emphasised that the quasi-LPV representation is valid globally with no restriction whatsoever to a neighbourhood of the equilibrium operating points. Hence, the class of quasi-LPV systems is indeed sufficiently rich to accommodate a wide range of gain-scheduling applications.

4.2 Small-gain LFT approaches

Packard (1994) and Apkarian & Gahinet (1995) consider LPV design approaches which are based on small gain theory. Packard (1994) considers discrete-time systems and this is extended to continuous-time systems by Apkarian & Gahinet (1995). In both cases consideration is confined to LPV systems which may be formulated as a linear time-invariant system enclosed by a feedback loop with time-varying parameter θ. Such systems are denoted parameter-dependent linear-fractional transformation (LFT) systems. Specifically, in the continuous-time case a generalised plant is considered

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \alpha \\ \mathbf{e} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \mathbf{B}_{11} & \mathbf{B}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \mathbf{B}_{21} & \mathbf{B}_{22} \\ \mathbf{C}_{11} & \mathbf{C}_{12} & \mathbf{D}_{11} & \mathbf{D}_{12} \\ \mathbf{C}_{21} & \mathbf{C}_{22} & \mathbf{D}_{21} & \mathbf{D}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \beta \\ \mathbf{d} \\ \mathbf{u} \end{bmatrix} \tag{27}$$

with feedback

$$\beta = \theta\,\alpha \tag{28}$$

where

$$\theta = \mathrm{blockdiag}\left(\theta_1 \mathbf{I}_{r_1} \quad \cdots \quad \theta_K \mathbf{I}_{r_K}\right) \tag{29}$$

and θ_i denotes the i[th] time-varying parameter, $\mathbf{I}_{r_i}$ denotes the $r_i \times r_i$ identity matrix and $r_i>1$ whenever the parameter θ_i is repeated. It is assumed that

$$|\theta_i| \le 1/\gamma \tag{30}$$

where γ is a positive constant. The generalised plant is depicted in block diagram form in figure 1. Consider the design of the output feedback controller

$$\begin{bmatrix} \dot{\mathbf{x}}_K \\ \alpha_K \\ \mathbf{u} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_K & \mathbf{B}_{K_1} & \mathbf{B}_{K_2} \\ \mathbf{C}_{K_1} & \mathbf{D}_{K_{22}} & \mathbf{D}_{K_{22}} \\ \mathbf{C}_{K_2} & \mathbf{D}_{K_{22}} & \mathbf{D}_{K_{22}} \end{bmatrix} \begin{bmatrix} \mathbf{x}_K \\ \beta_K \\ \mathbf{y} \end{bmatrix} \tag{31}$$

with parameter dependence

$$\beta_K = \theta\, \alpha_K \tag{32}$$

The requirement is to determine controller state-space matrices such that the closed-loop system is BIBO stable with output $\mathbf{e} \in L_2$ for all inputs $\mathbf{d} \in L_2$ and (for an initial state of zero)

$$\|\mathbf{e}\|_2 < \gamma \|\mathbf{d}\|_2 \qquad \forall\ \mathbf{d} \in L_2 \tag{33}$$

The feedback combination of the plant and controller is depicted in block diagram form in figure 2. It can be seen that the closed-loop system is a linear time-invariant system (comprising (27) and (31)) with time-varying feedback through the parameter θ. The small gain theorem states that a closed-loop system is stable provided the loop gain (as measured by an appropriate norm; the induced L_2 norm in the present case) is less than unity (see, for example, Desoer & Vidyasagar 1975). Hence, stability of the closed-loop system with parametric feedback θ is ensured provided the induced L_2 norm of $\mathbf{T}_\theta\theta$ is less than unity, where $\mathbf{T}_\theta$ denotes the linear time-invariant operator mapping $\begin{bmatrix} \alpha \\ \alpha_K \end{bmatrix}$ to $\begin{bmatrix} \beta \\ \beta_K \end{bmatrix}$. By (30), the induced L_2 norm of the parameter feedback is less than $1/\gamma$. Hence, a sufficient condition for closed-loop stability is that the induced L_2 norm of $\mathbf{T}_\theta$ is less than γ; that is, sufficient conditions for the solvability of the control design task are

$$\|\mathbf{T}\|_2 < \gamma\,,\ \ \|\mathbf{T}_\theta\|_2 < \gamma \tag{34}$$

where $\mathbf{T}$ denotes the linear time-invariant operator mapping $\mathbf{d}$ to $\mathbf{e}$. Owing to the diagonal structure (29) of the parameter feedback and the realness of the θ_i, the induced L_2 norm of θ is invariant with respect to certain classes of structured scalings. The conservativeness of the solvability conditions may be reduced by explicitly incorporating appropriate scalings into the solvability conditions (Apkarian & Gahinet 1995, Helmersson 1995a,b). For plants with the parameter-dependent LFT structure (27)-(28), the scaled small-gain solvability conditions can be reformulated equivalently as a numerically tractable convex feasibility problem with a finite number of constraints; that is, a finite number of linear matrix inequalities (Apkarian & Gahinet 1995).

It should be noted that the LFT structure plays a key role in obtaining a convex problem: the solvability conditions are non-convex for general parameter-dependent plants (Packard 1994, Apkarian & Gahinet 1995, Helmersson 1995b). The requirement that the plant possesses an LFT structure does not, in principle, seem to be particularly restrictive. LPV systems in which the state-space matrices are polynomial or rational functions of the parameters can be transformed into LFT form (see, for example, Helmersson 1995b, Belcastro 1998). However, this reformulation is non-trivial in general and may involve a considerable increase in the order of the system, since a large number of repeated parameter blocks can be required (see, for example, Helmersson 1995b), with a consequent increase in the order of the controller and the computational difficulty of the design problem.

The small-gain solvability conditions, (34), are valid not only when the parameter, θ, is an exogenous time-varying quantity but also in the more general case where the elements of the parameter, θ, might depend on the system state; that is, for quasi-LPV systems (Helmersson 1995b). However, depending on the similarity scalings employed, the scaled small-gain condition may be confined to LPV plants; that is, design approaches based on the scaled small-gain conditions may not therefore be applicable to general quasi-LPV systems (see, for example, Helmersson 1995b). Of course, for LPV systems, the scaled small-gain system is less conservative than the unscaled conditions. With the aim of further reducing conservativeness, Helmersson (1995b) briefly considers methods for incorporating knowledge of the rate of variation of the parameters into the design procedure. However, these lead to infinite dimensional solvability conditions and the numerical issues associated with the design of controllers to satisfy these conditions are not addressed.

The extension of the design approach to include uncertain parameters which are not available to the controller is conceptually straightforward using μ-type upper-bounding approaches but the associated solvability conditions are non-convex (even when the plant is in LFT form). A number of *ad hoc* iterative approaches to obtain approximate solutions to this non-convex problem are proposed in Apkarian & Gahinet (1995), Helmersson (1995a,b) and Spillman *et al.* (1996), but there is no guarantee of these procedures finding an adequate controller even when one exists.

4.3 Lyapunov-based LPV approaches

Following a similar approach to, for example, Isidori & Astolfi (1992), consider the nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}),\ \ \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}) \tag{35}$$

with $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, $|\mathbf{y}|_2 \le \kappa\left(|\mathbf{x}|_2 + |\mathbf{r}|_2\right)$, κ a positive constant. Let $V(\mathbf{x})$ be a continuously differentiable function such that

$$\alpha_1|\mathbf{x}|_2^{\,2} \le \mathrm{V}(\mathbf{x}) \le \alpha_2|\mathbf{x}|_2^{\,2} \tag{36}$$

$$\frac{\partial \mathrm{V}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{r}) + \mathbf{y}^{\mathrm{T}}\mathbf{y} - \gamma^2\mathbf{r}^{\mathrm{T}}\mathbf{r} \le -\alpha_3|\mathbf{x}|_2^{\,2} \qquad \forall \mathbf{r} \in \mathrm{L}_2 \tag{37}$$

where α_1, α_2, α_3, γ are positive constants. The condition, (36), ensures that V is positive definite and it should be noted that $\frac{\partial \mathrm{V}(\mathbf{x})}{\partial \mathbf{x}}\mathbf{F}(\mathbf{x},\mathbf{r})$ is just the time derivative of V along the solution trajectories of the nonlinear system (35). Hence, in the unforced case where **r** equals zero, (36)-(37) imply that V decreases uniformly along the solution trajectories of the nonlinear system; that is, (36)-(37) reduce to the standard Lyapunov conditions for exponential stability (see, for example, Khalil 1992 theorem 4.1). In the forced case where **r** is non-zero, by integrating the left hand side of (37) it follows that for initial condition $\mathbf{x}(0)=0$ the input and output of the nonlinear system satisfy

$$\int_0^{t_1} \mathbf{y}(t)^{\mathrm{T}}\mathbf{y}(t)dt \le \gamma^2 \int_0^{t_1} \mathbf{r}(t)^{\mathrm{T}}\mathbf{r}(t)dt \qquad \forall\ t_1 > 0 \tag{38}$$

or, equivalently,

$$\|\mathbf{y}\|_2 \le \gamma\|\mathbf{r}\|_2 \tag{39}$$

In other words, when there exists a Lyapunov-like function V satisfying (36)-(37) then the nonlinear system (35) is BIBO stable with induced L_2 norm γ. This type of result forms the basis of a number of LPV gain-scheduling approaches. In these approaches, various specific forms are assumed for the Lyapunov-like function, V. When the system considered is linear, it is sufficient to confine consideration to quadratic functions, V, of the form $\mathbf{x}^{\mathrm{T}}\mathbf{P}\mathbf{x}$ where **P** is positive definite. Of course, this does not extend to general nonlinear systems and any restriction to a particular class of functions, V, generally introduces a degree of conservativeness into the results obtained.

4.3.1 Quadratic Lyapunov function approaches

Shahruz & Behtash (1992, section 3.2) consider the design of a state feedback controller

$$\mathbf{r} = -\mathbf{K}(\theta)\mathbf{x} \tag{40}$$

for the LPV plant

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{r} \tag{41}$$

$$\mathbf{y} = \mathbf{C}(\theta)\mathbf{x} \tag{42}$$

where $\mathbf{A}(\bullet)$, $\mathbf{B}(\bullet)$, $\mathbf{C}(\bullet)$ are analytic functions of the parameter θ, and θ is a continuous bounded function of time. Necessary and sufficient conditions are derived for there to exist a state feedback controller such that $\mathbf{x}^{\mathrm{T}}\mathbf{x}$ is a Lyapunov function for the closed-loop system (in which case it follows immediately that the closed-loop system is exponentially stable for arbitrary parameter variations and so also for the quasi-LPV case where θ may depend on the state **x**). The existence conditions are constructive in the sense that suitable controller gains $\mathbf{K}(\theta)$ are explicitly obtained. However, the existence conditions require that a particular matrix function of θ satisfies a constraint for every allowable value of the parameter θ. Since θ takes values on a real interval, there are infinitely many (indeed uncountably many) constraints to be checked; that is, the existence conditions are in the form of a feasibility problem with infinitely many constraints. The existence conditions are not, therefore, readily testable. Similarly, the associated controller gain calculations require to be evaluated for every value of θ and thus are infinite dimensional. The former issue is not addressed by Shahruz & Behtash (1992, section 3.2). In response to the latter issue, a design procedure is presented whereby the controller gain matrix is calculated for a number of specific values of the parameter θ. The controller gain matrices for intermediate parameter values are then obtained by linear interpolation. Provided the existence conditions are satisfied (for all of θ) and the specific parameter values on which the interpolated controller design is based are sufficiently close together, it is shown that the resulting controller meets the requirements.

Becker *et al.* (1993,1994) generalise the approach of Shahruz & Behtash (1992, section 3.2) to the design of output feedback controllers

$$\dot{\mathbf{x}}_{\mathbf{K}} = \mathbf{A}_{\mathbf{K}}(\theta)\mathbf{x}_{\mathbf{K}} + \mathbf{B}_{\mathbf{K}}(\theta)\mathbf{y} \tag{43}$$

$$\mathbf{u} = \mathbf{C}_{\mathbf{K}}(\theta)\mathbf{x}_{\mathbf{K}} + \mathbf{D}_{\mathbf{K}}(\theta)\mathbf{y} \tag{44}$$

for plants

$$\dot{\mathbf{x}} = \mathbf{A}(\theta)\mathbf{x} + \mathbf{B}(\theta)\mathbf{u} \tag{45}$$

$$\mathbf{y} = \mathbf{C}(\theta)\mathbf{x} + \mathbf{D}(\theta)\mathbf{u} \tag{46}$$

and quadratic Lyapunov functions of the form $\mathbf{x}_{\mathbf{c}}^{\mathrm{T}}\mathbf{P}\mathbf{x}_{\mathbf{c}}$, where **P** is a positive definite symmetric matrix and $\mathbf{x}_{\mathbf{c}}$ is the state $[\mathbf{x}^{\mathrm{T}}\ \mathbf{x}_{\mathbf{K}}^{\mathrm{T}}]^{\mathrm{T}}$ of the closed-loop system. The plant matrices are required to be continuous bounded functions of the parameter, θ, and the parameters are assumed to be piecewise continuous functions of time. Becker *et al.* (1993) derive necessary and sufficient conditions for there to exist a controller (43)-(44) such that $\mathbf{x}_{\mathbf{c}}^{\mathrm{T}}\mathbf{P}\mathbf{x}_{\mathbf{c}}$ is a Lyapunov function of the closed-loop system (the so-called quadratic stabilisation problem) and a Youla-type parameterisation of all such controllers is presented. Similarly to Shahruz & Behtash (1992), the existence conditions take the form of a feasibility problem with infinitely many constraints. Apkarian *et al.* (1995) extend the results of Becker *et al.* (1993,1994) to discrete time systems.

In addition, Becker *et al.* (1993,1994) consider incorporating an induced L_2-norm performance requirement into the controller design approach. Specifically, a generalised plant is considered

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \mathbf{e_1} \\ \mathbf{e_2} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{A}(\theta) & \mathbf{B_1}(\theta) & \mathbf{0} & \mathbf{B_2}(\theta) \\ \mathbf{C_1}(\theta) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{C_2}(\theta) & \mathbf{0} & \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{d_1} \\ \mathbf{d_2} \\ \mathbf{u} \end{bmatrix} \tag{47}$$

where $\mathbf{x}$ is the state, $\mathbf{y}$ is the measured output available to the controller and $\mathbf{u}$ is the control input to the plant; $\mathbf{d_1}$, $\mathbf{d_2}$ are unmeasured disturbance inputs and $\mathbf{e_1}$, $\mathbf{e_2}$ are outputs which indicate performance. The requirement is to determine an output feedback controller, (43)-(44), such that $\mathbf{x_c}^T\mathbf{P}\mathbf{x_c}$ is a Lyapunov function (when the input $\mathbf{d_1}$, $\mathbf{d_2}$ equal zero *i.e*. the unforced case) of the closed-loop LPV system and, in addition,

$$\|\mathbf{e}\|_2 < \gamma\|\mathbf{d}\|_2 \tag{48}$$

where $\mathbf{e} = [\mathbf{e_1}^T\ \mathbf{e_2}^T]^T$, $\mathbf{d} = [\mathbf{d_1}^T\ \mathbf{d_2}^T]^T$, γ is a positive constant. Conditions for the solvability of this control problem are derived and once again these take the form of a feasibility problem with infinitely many constraints. Apkarian *et al.* (1995) extend the results to discrete time systems. It should be noted that the foregoing approaches are derived in the context of LPV systems where the parameter θ is an exogenous time-varying quantity which is independent of the state $\mathbf{x}$ of the system. However, the controllers obtained are valid for arbitrary variations in θ; that is, the results are also directly applicable to quasi-LPV systems where the parameter θ may depend on the state of the system.

A primary practical difficulty with the foregoing approaches is that the solvability conditions involve an infinite number of constraints and so the task of determining a controller which satisfies these conditions is intractable numerically. This arises because a constraint must be satisfied for every allowable parameter value which leads to uncountably many constraints since there is a continuum of parameter values. In order to address this problem, Becker *et al*. (1993) propose an approximate, *ad hoc* approach whereby the parameter space is divided into a fine grid and a controller is designed which satisfies the solvability conditions at a finite number of parameter values. However, it is noted that there appears to be little guidance as to how to perform the gridding. Moreover, for a particular grid spacing, the number of grid points required grows extremely rapidly as the number of parameters increases. Consequently, despite the relative efficiency of the available numerical algorithms for solving linear matrix inequalities, the utility of this approach with present computing facilities is strictly limited to systems with a small number of parameters (less than three or four) (Becker *et al.* 1993).

Whilst there is, in general, no systematic method for avoiding such gridding, simplifications are possible (possibly at the cost of additional conservativeness) for some specific classes of plant. For affine LPV systems with parameter values belonging to a convex polytope (see Appendix A), the solvability conditions reduce to a feasibility problem with a *finite* number of constraints (Becker *et al.* (1993), Apkarian *et al.* (1995)). Specifically, it is sufficient to evaluate the constraints associated with the vertices of the polytope of parameter values since this ensures satisfaction of the constraints for every parameter value within the polytope. When the parameter set is not convex, this approach can still be employed when the actual parameter set is contained within some larger convex polytope. Of course, an additional degree of conservativeness (which may be quite considerable when there is a large number of parameters) is introduced since the control design accommodates combinations of parameter values which cannot occur on the actual plant. Confining consideration to the class of affine LPV systems also introduces considerable conservativeness in general; for example, any nonlinear dependence between parameters cannot easily be modelled and such parameters are treated in the design procedure as varying independently.

In addition to the foregoing practical difficulties, the approaches considered are inherently conservative as a result of their requirement that the closed-loop system has a quadratic Lyapunov function. In general, there may exist a controller which stabilises a plant (with the closed-loop system having a non-quadratic Lyapunov function) but there may be no controller which ensures that the closed-loop system has a quadratic Lyapunov function (an example of such a system is given by Wu *et al.* 1995). At present, this important issue remains largely open although tractable conditions for the existence of a common Lyapunov function are currently being actively researched (see for example Shorten & Narendra 1998).

4.3.2 Parameter-dependent Lyapunov function approaches

One of the earliest Lyapunov-based approaches proposed in the context of gain-scheduling is that of Shamma (1988, section 5.2), Shamma & Athans (1992). A class of inverting controllers is considered which asymptotically substitute target loop dynamics corresponding to a time-varying Kalman filter for a class of linear time-varying plants. Whilst a Kalman filter is quite a general representation in a linear time-invariant context, in the time-varying case the Kalman filter is considerably more restrictive. The time-varying Kalman filter dynamics are nonlinear but are selected to ensure the existence of a particular time-varying quadratic Lyapunov function. Whilst the Kalman filter target dynamics support rapidly time-varying dynamics, in order for these dynamics to be recovered by the closed-loop controller/plant combination it is necessary to ensure the stability of an auxiliary time-varying system (Shamma 1988). Stability analysis of the latter system is non-trivial and may require the introduction of a slow variation condition, despite the support of the target dynamics for rapid variations. Shamma (1988, section 5.3) extends the approach to nonlinear plants which can be transformed into quasi-LPV form.

More recently, motivated by the conservativeness associated with the LPV methods described in section 4.3.1, Wu *et al.* (1995), Apkarian & Adams (1997, 1998) consider the design of controllers for LPV plants to ensure that the (unforced) closed-loop system has a *parameter-dependent* Lyapunov function of the form $\mathbf{x_c}^T\mathbf{P}(\theta)\mathbf{x_c}$, where $\mathbf{P}(\theta)$ is positive definite symmetric and $\mathbf{x_c}$ is the state of the closed-loop system. Specifically, for the generalised LPV plant, (47), the requirement is to determine an output feedback controller

$$\dot{\mathbf{x}}_\mathbf{K}=\mathbf{A_K}(\theta,\ \dot{\theta})\mathbf{x_K}+\mathbf{B_K}(\theta,\ \dot{\theta})\mathbf{y} \tag{49}$$

$$\mathbf{u}=\mathbf{C_K}(\theta,\ \dot{\theta})\mathbf{x_K} + \mathbf{D_K}(\theta,\ \dot{\theta})\mathbf{y} \tag{50}$$

such that $\mathbf{x_c}^T\,\mathbf{P}(\theta)\mathbf{x_c}$ is a Lyapunov function of the (unforced) closed-loop LPV system and, in addition,

$$\|\mathbf{e}\|_2 < \gamma\|\mathbf{d}\|_2 \tag{51}$$

where $\mathbf{e} = [\mathbf{e_1}^T\ \mathbf{e_2}^T]^T$, $\mathbf{d} = [\mathbf{d_1}^T\ \mathbf{d_2}^T]^T$, γ is a positive constant . The time derivative $\dot{\theta}$ is assumed to be well-defined at all times with the elements of $\dot{\theta}$ uniformly bounded. It should be noted that, in contrast to the situation considered in section 4.3.1, the controller matrices now depend on both the parameter θ and its time derivative $\dot{\theta}$. In addition, whilst the parameter could previously vary arbitrarily rapidly, it is now assumed to be bounded (hence it is not possible for θ to depend on the state $\mathbf{x}$ of the system without restricting the input and the initial conditions of the state to ensure slow variation; there do not appear to be any results in the LPV literature regarding this situation). Conditions for the solvability of this control problem are derived and these take the form of an infinite dimensional feasibility problem with infinitely many constraints.

Apkarian & Adams (1998) note that, since the parameter derivative $\dot{\theta}$ frequently can neither be measured nor estimated easily during system operation, controllers of the form (49)-(50) are impractical. Whilst this cannot be resolved without loss of generality, the dependence of the controller on $\dot{\theta}$ can be removed by further restricting consideration to a sub-class of Lyapunov functions (Apkarian & Adams 1998). Further practical difficulties are associated with the infinite number of constraints which must be satisfied to meet the solvability conditions and with the infinite dimensional nature of $\mathbf{P}(\bullet)$ ($\mathbf{P}(\bullet)$ is a continuous matrix function rather than a constant matrix as in the approaches of section 4.3.1). With regard to the former, *ad hoc* gridding approaches may be used as in section 4.3.1 to obtain an approximate solution but these are numerically intractable except when there are a small number of parameters (Wu *et al.* 1995, Apkarian & Adams 1998). Alternatively, as in section 4.3.1, the constraints reduce to a finite number for the specific class of affine (or polynomial, Apkarian & Adams (1997)) LPV plants with parameters belonging to a convex polytope. With regard to the latter (infinite dimensional nature of $\mathbf{P}(\bullet)$), Wu *et al.* (1995) propose a Raleigh-Ritz type of approach whereby the functions relating to $\mathbf{P}(\bullet)$ in the solvability conditions are assumed to be of the form $\mathbf{X}(\theta) = \sum_i \alpha_i(\theta)\mathbf{X}_i$, where the $\mathbf{X}_i$ are constants which are selected to satisfy the solvability conditions and the $\alpha_i(\bullet)$ are appropriate basis functions which are assumed *a priori* known. Apkarian & Adams (1998) adopt a similar approach. Whilst the choice of basis functions can have a considerable influence on the conservativeness of the results obtained (Lim & How 1997), Wu *et al.* (1995) note that there is a complete lack of guidance provided by the theory as to how the basis functions, $\alpha_i(\bullet)$, should be selected. More recently, Lim & How (1997) consider an alternative approach which utilises parameter-dependent quadratic Lyapunov functions where $\mathbf{P}(\theta)$ is a piecewise affine function of the parameter θ. Provided the parameter values belong to a convex polytope, the plant is also piecewise affine, and a finite number of piecewise elements are used in the Lyapunov function and the plant, this leads to a finite dimensional feasibility problem with a finite number of constraints. However, whilst any continuous function can be approximated by a piecewise affine function with sufficiently many piecewise elements, the computational burden of the LPV design approach increases rapidly with the number of piecewise elements. Nevertheless, this seems a promising approach for situations where it is possible to employ a small number of piecewise elements.

In addition to the foregoing practical issues, there are a number of fundamental theoretical issues relating to parameter-dependent LPV methods which remain to be addressed. It is known that the class of quadratic Lyapunov functions, $\mathbf{x}^T\mathbf{P}\mathbf{x}$, is necessary and sufficient for the stability analysis of linear time-invariant systems (see, for example, Khalil 1992). Furthermore, the class of time-varying quadratic Lyapunov functions, $\mathbf{x}^T\mathbf{P}(t)\mathbf{x}$, is necessary and sufficient for the stability analysis of linear time-varying systems (see, for example, Khalil 1992 Theorems 4.1 and 4.3). While the class of parameter-dependent quadratic Lyapunov functions therefore seems a natural choice for LPV systems (although not necessarily for quasi-LPV systems), the structure implied by the parameter dependence introduces the possibility of conservativeness and there do not appear to be any existence (converse Lyapunov) results in the literature which establish a theoretical basis for the use of parameter-dependent quadratic Lyapunov functions or which support the restriction to the sub-class of such Lyapunov functions for which the LPV design methods lead to a controller which does not depend on $\dot{\theta}$. Hence, whilst the class of parameter-dependent quadratic Lyapunov functions, $\mathbf{x_c}^T\mathbf{P}(\theta)\mathbf{x_c}$, is certainly more general than the class of quadratic Lyapunov functions employed in the approaches of section 4.3.1, there may exist a controller which stabilises a plant (with the closed-loop system having some nonlinear Lyapunov function) but no controller which ensures that the closed-loop system has a parameter-dependent quadratic Lyapunov function. This possibility arises from the difficulty of choosing a sufficiently general class of Lyapunov function and is, of course, common to almost all constructive Lyapunov methods.

Also, it is noted that the existing parameter-dependent Lyapunov function approaches are confined to situations where the parameter θ is an exogenous quantity which may depend on time but is strictly independent of the state of the system (in marked contrast to the small-gain approaches described in section 4.2 and the common quadratic Lyapunov function approaches of section 4.3.1). As noted in section 4.1, this is a restrictive requirement in general and the parameter-dependent Lyapunov function approaches, in their present form, exclude many practical applications (including, for example, flight control applications where the parameters might typically depend on airspeed and/or angle of incidence, both of which are related to the system state). Fortunately, this is probably not a fundamental limitation since by restricting the input and the initial conditions of the state it is possible to ensure slow variation of the state of the closed-loop system (provided that the system is stable in an appropriate sense) and so of any dependent parameters in θ. However, this leads to a coupled design problem since the required restrictions on the input and initial conditions are themselves dependent on the choice of controller. For example, the restrictions on the input and initial conditions, required to ensure that the magnitude of $\dot{\theta}$ is less than a particular value, might be much stronger for an aggressive controller design than for one which generates only weak control action. Whilst this coupling appears to be neglected, for example, in the quasi-LPV design study in Apkarian & Adams (1998), it is emphasised that the coupling has, in general, a direct implication for the stability characteristics of the closed-loop system. Despite its practical importance, there do not appear to be any theoretical results in the LPV literature regarding the more general quasi-LPV situation.

5. Outlook

While the literature relating to gain-scheduling methods extends back at least thirty years, in the research community there has been a considerable upsurge in interest in these methods in the last ten years with many new and interesting approaches proposed. It might be suggested that one factor linking these methods is their shared aim of relaxing the restrictions associated with classical gain-scheduling approaches. In particular, relaxing

- the restriction to near equilibrium operation that arises from the use of only equilibrium information (namely, the equilibrium linearisations of the plant) for control design purposes.
- the slow variation conditions associated with ensuring that the overall system does not evolve between operating regions in too rapid a manner.

With regard to the former, new representations based, for example, on fuzzy/neural approaches, linear/quasi-linear parameter-varying formulations and off-equilibrium linearisation appear to address this issue with a good degree of success. In contrast, relaxation of the latter slow variation restriction is still very much an open question. This is perhaps unsurprising since the whole issue of obtaining tight, necessary and sufficient, conditions for the stability of nonlinear systems remains unresolved. (Related to stability analysis, it is worth noting that the literature relating to the robustness analysis of nonlinear, and in particular gain-scheduled, controllers remains very immature). The conditions under which arbitrary rates of variation are permitted in classical frozen-time/frozen-input theory are clearly conservative and recently proposed small gain/common quadratic Lyapunov function criteria seem to similarly suffer from a high degree of conservativeness; the conservativeness, or otherwise, of parameter-dependent quadratic Lyapunov criteria largely remains to be established. Despite this lack of tight theoretical results, it should be noted that gain-scheduled controllers are often observed to perform well even when slow variation, or other, conditions are violated suggesting that further research on this topic is certainly justified.

A number of technical issues relating to particular methodologies are highlighted in the main body of the paper and not repeated here. Other important open issues of general interest associated with gain-scheduling approaches, both classical and modern, include the pressing requirement for soundly-based interpolation techniques (these are needed not only for divide and conquer design methods but also to properly support the gridding phase commonly present in LPV methods) and for methods to assist in the selection of an appropriate scheduling variable and, related to this, an appropriate controller realisation. In many ways these are, in fact, modelling issues. It seems clear that interpolation approaches should reflect the plant characteristics at intermediate operating points and, similarly, that the choice of scheduling variable should reflect the nonlinear structure of the plant in some appropriate sense. The requirement is therefore for suitable plant information and, in particular, system representations which provide specific support for control design. This area is largely undeveloped at present, despite initial work by a number of researchers (particularly in the neural and fuzzy communities).

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Appendix A - Notation

The notation employed is standard. The Euclidean or 2-norm of a vector $\sigma\in\Re^n$ is denoted $|\sigma|_2$ and defined by $|\sigma|_2{}^2=\sigma^T\sigma$. A function $\mathbf{s}(t):[0,\infty)\rightarrow\Re^n$ has L_2 norm $\|\mathbf{s}\|_2$ defined by $\|\mathbf{s}\|_2{}^2=\int_0^\infty \mathbf{s}^T(\tau)\mathbf{s}(\tau)d\tau$. L_2 denotes the class of functions with bounded L_2 norm; that is, the class of square integrable functions. Letting $\mathbf{s}=\mathbf{Tr}$, the induced L_2 norm of the operator $\mathbf{T}$ is $\sup_{\mathbf{r}\in L_2} \frac{\|\mathbf{s}\|_2}{\|\mathbf{r}\|_2}$. With a standard abuse of notation, $\|\mathbf{T}\|_2$ is used to denote the induced 2-norm. The time derivative $d\mathbf{x}/dt$ of a vector $\mathbf{x}(t)\in\Re^n$ is denoted $\dot{\mathbf{x}}$. In situations where any choice of p-norm, $p\in[1,\infty]$, is admissible, the notation $|\sigma|$ is employed. Similarly, $\|\mathbf{s}\|$ indicates that any choice of L_p norm, $p\in[1,\infty]$, is admissible. The derivative $\nabla_\mathbf{x}\mathbf{f}(\mathbf{x})$ of a function $\mathbf{f}:\Re^n\rightarrow\Re^m$ denotes the usual Jacobian $\begin{bmatrix} \frac{d\mathbf{f}_1}{d\mathbf{x}_1} & \cdots & \frac{d\mathbf{f}_m}{d\mathbf{x}_1} \\ & \ddots & \\ \frac{d\mathbf{f}_1}{d\mathbf{x}_n} & \cdots & \frac{d\mathbf{f}_m}{d\mathbf{x}_n} \end{bmatrix}$ where $\mathbf{x}_i$ is the i^{th} element of the vector $\mathbf{x}$ and similarly for $\mathbf{f}_i$. A convex polytope is defined as a convex set bounded by flat surfaces; more precisely, a convex polytope is defined as the convex hull of a finite number, r, of matrices, N_i, with the same dimensions; that is, $\left\{\sum_{i=1}^{r}\alpha_i N_i : \alpha_i \geq 0, \sum_{i=1}^{r}\alpha_i = 1\right\}$ (see, for example, Apkarian *et al.* 1995). Every point in a convex polytope is equal to a linear combination of the vertices, N_i. An LPV system is affine when the state space matrices $\mathbf{A}(\theta)$, $\mathbf{B}(\theta)$, $\mathbf{C}(\theta)$, $\mathbf{D}(\theta)$ depend affinely on θ; that is, $\mathbf{A}(\theta) = \mathbf{A}_o + \sum_{i=1}^{n}\beta_i(\theta)\mathbf{A}_i$ and similarly for $\mathbf{B}(\theta)$, $\mathbf{C}(\theta)$ and $\mathbf{D}(\theta)$. When the allowable parameter values of an affine LPV system are restricted to lie within a convex polytope in the parameter space, it follows from the foregoing definitions that the state-space matrices $\mathbf{A}(\theta)$, $\mathbf{B}(\theta)$, $\mathbf{C}(\theta)$, $\mathbf{D}(\theta)$ lie within a convex polytope of matrices (see, for example, Apkarian *et al.* 1995). The following stability definitions are employed.

Exponential Stability (see, for example, Khalil 1992 p168)

An unforced dynamic system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, t)$$

where $\mathbf{x} \in \Re^n$, is locally exponentially stable if there exist strictly positive constants κ, a and c such that

$$|\mathbf{x}(t)| \leq \kappa e^{-a(t-t_o)}\,|\mathbf{x}(t_o)| \quad \forall t \geq t_o \geq 0,\ |\mathbf{x}(t_o)| < c$$

The system is globally exponentially stable if this inequality is satisfied with c unbounded.

Bounded Input-Bounded Output (BIBO) Stability

Various definitions of BIBO stability, differing in technical details, are used in the literature; for example, Desoer & Vidyasagar (1975), Vidyasagar & Vannelli (1982) assume that the initial condition of the state is always zero whereas Sontag (1989) and Khalil (1992) permit non-zero initial conditions. Rather than specifying a plethora of BIBO stability definitions, the following definition is adopted since the stability results discussed in this paper relating to gain-scheduling can be readily formulated in terms of this definition. A forced dynamic system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}, t),\quad \mathbf{y} = \mathbf{G}(\mathbf{x}, \mathbf{r}, t)$$

where $\mathbf{r} \in \Re^m$, $\mathbf{y} \in \Re^p$, $\mathbf{x} \in \Re^n$, is locally BIBO stable if there exist positive constants c and d such that, for $\mathbf{r}\in L_2$,

$$|\mathbf{y}(t)| \leq \gamma(\sup_{t\geq t_o}|\mathbf{r}(t)|) + \beta(|\mathbf{x}(t_o)|, t) \qquad \forall\, t \geq t_o > 0,\ |\mathbf{x}(t_o)| < c,\ |\mathbf{r}(t)| < d$$

where $\gamma(\sup_{t\geq t_o}|\mathbf{r}(t)|)$ is a class K_∞ function (an unbounded strictly increasing function of $\sup_{t\geq t_o}|\mathbf{r}(t)|$) and $\beta(|\mathbf{x}(t_o)|, t)$ is a class *KL* function (β is strictly increasing with respect to $|\mathbf{x}(t_o)|$ for each fixed t and zero when $\mathbf{x}(t_o)|$ is zero, and β is strictly decreasing with respect to t for each fixed $|\mathbf{x}(t_o)|$ and $\beta\rightarrow 0$ as $t\rightarrow\infty$). The system is globally BIBO stable if this inequality is satisfied with c and d unbounded. Unless otherwise stated, references to BIBO stability hereafter denote systems where $\gamma(\sup_{t\geq t_o}|\mathbf{r}(t)|)$ is of the form $\gamma\sup_{t\geq t_o}|\mathbf{r}(t)|$, with $\gamma<\infty$, and $\beta(|\mathbf{x}(t_o)|, t)$ is of the exponential form $\kappa e^{-a(t-t_o)}|\mathbf{x}(t_o)|$, κ, a>0. It is noted that such systems are exponentially stable when the input, $\mathbf{r}$, is zero.

Appendix B – Series expansion linearisation about a single trajectory or equilibrium point

The literature considering the relationship between the local stability of the nonlinear system (1) that of the series expansion linearisation, (2)-(3), is extensive yet very fragmented and so it is necessary to bring together many separate results in order to support the summary in section 2.1.

It follows from standard Lyapunov theory that when $\delta\mathbf{r}$ is zero (*i.e.* the unforced case) the nonlinear state dynamics of (1) are locally exponentially stable if and only if the unforced linear time-varying system

$$\delta\dot{\hat{\mathbf{x}}} = \nabla_x\mathbf{F}(\tilde{\mathbf{x}}, \tilde{\mathbf{r}})\delta\hat{\mathbf{x}} \qquad (52)$$

is stable (see, for example, Khalil 1992 p184). When $\delta\mathbf{r}$ is non-zero, the nonlinear system, (1), is locally BIBO stable provided (52) is stable, $\delta\mathbf{x}$ is initially zero, $\delta\mathbf{r}$ is sufficiently small and the derivatives $\nabla_r\mathbf{F}$, $\nabla_x\mathbf{G}$, $\nabla_r\mathbf{G}$ are uniformly bounded (Vidyasagar & Vannelli 1982, Vidyasagar 1993 section 6). Since this holds for all time, it is straightforward to show that the initial conditions need not be restricted to be zero and the result may be extended to encompass other initial conditions, provided that they are sufficiently close to the origin (see, for example, Khalil 1992 p216). In the special case when the system, (52), is linear time-invariant, simple necessary and sufficient conditions for its stability are well-known (see, for example, Vidyasagar 1993). However, in the time-varying case, the stability analysis of (52) is, in general, not so straightforward.

Consider the unforced linear time-varying system,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \qquad (53)$$

where $\mathbf{x} \in \Re^n$. Let the constant matrix, $\mathbf{A}_\tau$, denote the value of $\mathbf{A}(t)$ at time, τ. Assume that $\mathbf{A}(\cdot)$ is bounded, differentiable and the eigenvalues of $\mathbf{A}_\tau$ lie in the left-half complex plane and are uniformly bounded away from the imaginary axis for every value of τ, then the linear time-varying system, (53), is globally exponentially stable provided $\sup_{t\geq 0} |\dot{\mathbf{A}}(t)|$ is sufficiently small (asymptotic stability initially derived by Rosenbrock (1963) and subsequently sharpened to exponential stability by Desoer (1969)). It should be noted that this result only establishes a sufficient condition for stability: when the rate of variation of $\mathbf{A}(t)$ is not sufficiently slow, Rosenbrock (1963) gives an example where the system is unstable. The requirement that the eigenvalues of $\mathbf{A}_\tau$ are uniformly bounded away from the imaginary axis may be relaxed to permit the eigenvalues to tend asymptotically to the imaginary axis but then exponential stability of the system is replaced by weaker asymptotic stability (Amato *et al.* 1993). The differentiability condition on $\mathbf{A}(\cdot)$ may be relaxed to a requirement for Lipschitz continuity or the restriction on $\sup_{t\geq 0} |\dot{\mathbf{A}}(t)|$ may be replaced by a restriction on the moving average, $\frac{1}{T}\int_t^{t+T} |\dot{\mathbf{A}}(s)|\,ds$ (see, for example, Ilchmann *et al.* 1987, Khalil 1992 section 4.5) or a restriction on $\sup_{t\geq 0} \min\left(|\dot{\theta}(t)|, |\ddot{\theta}(t)|\right)$ (when $\mathbf{A}(t)=\mathbf{A}(\theta(t))$, Guo & Rugh 1995). Furthermore, the requirement for the continuity of $\mathbf{A}(t)$ may be relaxed provided that any discontinuities in $\mathbf{A}(t)$ occur sufficiently infrequently (Zhang 1993, Morse 1995 p97). However, it should be noted that the available results relating the stability of the nonlinear system, (1), to the stability of the unforced system, (52), require at least Lipschitz continuity of $\nabla_x\mathbf{F}$. Although the foregoing results are for continuous time systems, similar results may also be derived for discrete linear time-varying systems (see, for example, Desoer 1970, Dahleh & Dahleh 1991, Amato *et al.* 1993).

The frozen-time type of analysis is extended by Barman (1973) to unforced smoothly-nonlinear time-varying systems with a *single* equilibrium operating point (see also Desoer & Vidyasagar 1975 section 4.8, Vidyasagar 1993 section 5.8.2). By applying this result to the perturbations

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(\hat{\mathbf{x}}, \tau) \qquad (54)$$

of the family of linear time invariant systems,

$$\dot{\mathbf{x}} = \mathbf{A}_\tau\mathbf{x} \qquad (55)$$

where the frozen-time nonlinear systems, (54), are uniformly exponentially stable and $\mathbf{F}(0, \tau)$ is zero (so that the equilibrium point is uniformly the origin), it follows immediately that the time-varying perturbed system,

$$\dot{\hat{\mathbf{x}}} = \mathbf{F}(\hat{\mathbf{x}}, t) \qquad (56)$$

is also exponentially stable provided the rate of time variation is sufficiently slow (in an appropriate sense). Consequently, provided the rate of variation is sufficiently slow, the linear time-varying system, (53), inherits the stability robustness of the family of linear time invariant systems, (55), to smooth nonlinear dynamic perturbations of arbitrary finite dimension which preserve the origin as the equilibrium point. It is also shown by Shamma & Athans (1991) that, provided the rate of variation is sufficiently slow, the linear time-varying system, (53), inherits the stability robustness of the linear time-invariant systems, (55), to infinite dimensional linear time-invariant perturbations in the dynamics. In addition, it follows trivially from the analysis of Leith &Leithead (1998b) that, provided the rate of variation is sufficiently slow, (53) inherits the worst-case induced L_∞ norm of family, (55) and so the robustness of (55) with respect to perturbations in L_∞ (which includes a wide class of non-smooth and infinite dimensional nonlinear perturbations).

Appendix C – LPV formulations

Pseudo-linear systems

The series expansion linearisation of a nonlinear system is obtained by truncating the Taylor series expansion of $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ after the first two terms. Owing to this truncation, the series expansion linearisation representation is only valid locally to a specific trajectory or equilibrium operating point. Provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are differentiable sufficiently many times, the neighbourhood within which the series expansion representation is valid can be enlarged by truncating the series expansion later thereby retaining more higher-order terms. For example, Banks & Al-Jurani (1996) consider the unforced nonlinear system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{57}$$

where $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot)$ is analytic, and $\mathbf{F}(0) = 0$. By retaining infinitely many higher-order terms in the Taylor series expansion of $\mathbf{F}(\cdot)$ about the origin, the nonlinear system (57) may be reformulated equivalently as the quasi-LPV system (referred to be Banks & Al-Jurani (1996) as a pseudo-linear system)

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} \tag{58}$$

provided the initial condition of $\mathbf{x}$ is restricted such that the components of $\mathbf{x}(t)$ are uniformly bounded. Whilst Banks & Al-Jurani (1996) restrict consideration to unforced systems, the extension to forced systems, (1), is straightforward provided $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic and $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ (see, for example, Helmersson 1995b chapter 10). The assumption that $\mathbf{F}(0,0)=0=\mathbf{G}(0,0)$ is not restrictive, provided the system has at least one equilibrium operating point, since this requirement can always be satisfied by adding/subtracting a constant from the state, input and output. The requirement that $\mathbf{F}(\cdot,\cdot)$ and $\mathbf{G}(\cdot,\cdot)$ are analytic (and therefore infinitely differentiable) is rather stronger. Nevertheless, the main difficulties with this approach are essentially practical in nature. Owing to the difficulty of evaluating the higher-order derivatives of a non-linear function and the sensitivity of polynomial series expansions to errors in the coefficients of the higher order terms, the infinite series approach of Banks & Al Jurani (1996) is impractical and it is almost always necessary, in practice, to employ a truncated series with only a finite number of terms. Indeed, it is frequently necessary to truncate the infinite series after only a relatively small number of terms, particularly for high-order systems with a large number of states, inputs and outputs. Hence, the quasi-LPV representation obtained by this method is, in practice, only valid within a neighbourhood about the origin. Whilst this neighbourhood subsumes that within which the series expansion linearisation is valid, it may nevertheless still be small.

Reformulation by mean value theorem

The mean value theorem can be employed to reformulate a general nonlinear system in quasi-LPV form. Consider the nonlinear system,

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}, \mathbf{r}) \tag{59}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{x} \in \Re^n$, $\mathbf{F}(\cdot,\cdot)$ is differentiable with bounded, Lipschitz continuous derivatives. It follows from the mean value theorem (see, for example, Boyd *et al.* 1994) that

$$\mathbf{c}^T\left(\mathbf{F}(\mathbf{x},\mathbf{r}) - \mathbf{F}(\bar{\mathbf{x}},\bar{\mathbf{r}})\right) = \mathbf{c}^T\nabla_x\mathbf{F}(\mathbf{z}_\mathbf{x},\mathbf{z}_\mathbf{r})\left(\mathbf{x}-\bar{\mathbf{x}}\right) + \mathbf{c}^T\nabla_\mathbf{r}\mathbf{F}(\mathbf{z}_\mathbf{x},\mathbf{z}_\mathbf{r})\left(\mathbf{r}-\bar{\mathbf{r}}\right) \tag{60}$$

where $\mathbf{c}^T \in \Re^n$ and $(\mathbf{z_x},\mathbf{z_r})$ is a point lying on the line segment in $\Re^n \times \Re^m$ joining $(\mathbf{x},\mathbf{r})$ and $(\bar{\mathbf{x}},\bar{\mathbf{r}})$. Hence, assuming without loss of generality that $\mathbf{F}(0,0)=0$, then

$$\mathbf{c}^T\mathbf{F}(\mathbf{x},\mathbf{r}) = \mathbf{c}^T\nabla_\mathbf{x}\mathbf{F}(\mathbf{z}_\mathbf{x},\mathbf{z}_\mathbf{r})\mathbf{x} + \mathbf{c}^T\nabla_\mathbf{r}\mathbf{F}(\mathbf{z}_\mathbf{x},\mathbf{z}_\mathbf{r})\mathbf{r} \tag{61}$$

Since $\mathbf{c}$ can take any value in $\Re^n$, it follows that

$$\dot{\mathbf{x}} = \begin{bmatrix} \left(\nabla_\mathbf{x}\mathbf{F}(\mathbf{z}_{\mathbf{x}_1},\mathbf{z}_{\mathbf{r}_1})\right)_1 \\ \vdots \\ \left(\nabla_\mathbf{x}\mathbf{F}(\mathbf{z}_{\mathbf{x}_n},\mathbf{z}_{\mathbf{r}_n})\right)_n \end{bmatrix}\mathbf{x} + \begin{bmatrix} \left(\nabla_\mathbf{r}\mathbf{F}(\mathbf{z}_{\mathbf{x}_1},\mathbf{z}_{\mathbf{r}_1})\right)_1 \\ \vdots \\ \left(\nabla_\mathbf{r}\mathbf{F}(\mathbf{z}_{\mathbf{x}_n},\mathbf{z}_{\mathbf{r}_n})\right)_n \end{bmatrix}\mathbf{r} \tag{62}$$

where $\left(\nabla_\mathbf{x}\mathbf{F}(\mathbf{z}_{\mathbf{x}_i},\mathbf{z}_{\mathbf{r}_i})\right)_i$ denotes the i[th] row of $\nabla_\mathbf{x}\mathbf{F}(\mathbf{z}_{\mathbf{x}_i},\mathbf{z}_{\mathbf{r}_i})$ and the $(\mathbf{z}_{\mathbf{x}_i},\mathbf{z}_{\mathbf{r}_i})$, i=1,..n, are points which lie on the line segment in $\Re^n \times \Re^m$ from $(\mathbf{x},\mathbf{r})$ to the origin. It should be noted that the points $(\mathbf{z}_{\mathbf{x}_i},\mathbf{z}_{\mathbf{r}_i})$, i=1,..n, strongly depend, in general, on the value of $(\mathbf{x},\mathbf{r})$ and so vary as the solution $(\mathbf{x}(t),\mathbf{r}(t))$ to the system evolves. Evidently, the dynamics, (34), are in quasi-LPV form with parameter $\theta = \left[\left[\mathbf{z}_{\mathbf{x}_1}{}^T \quad \mathbf{z}_{\mathbf{r}_1}{}^T\right] \ \cdots \ \left[\mathbf{z}_{\mathbf{x}_n}{}^T \quad \mathbf{z}_{\mathbf{r}_n}{}^T\right]\right]^T$.

Unfortunately, whilst applicable to a general class of systems, the reformulation, (34), is essentially an existence result and, in particular, provides little insight regarding a general means for determining the specific value of θ associated with an operating point, $(\mathbf{x},\mathbf{r})$. LPV control synthesis techniques generally require that a measurement or estimate of the varying parameter, θ, is available to the controller. In addition, owing to the conservativeness of controllers designed to accommodate arbitrary rates of parameter variation, it is often required that an upper bound can be placed on $\dot{\theta}$. Hence, it is necessary to explicitly establish a mapping from

$(\mathbf{x},\mathbf{r})$ to θ; that is, from $(\mathbf{x},\mathbf{r})$ to $\left[\begin{bmatrix}\mathbf{z}_{x_1}{}^T & \mathbf{z}_{\mathbf{r}_1}{}^T\end{bmatrix} \cdots \begin{bmatrix}\mathbf{z}_{x_n}{}^T & \mathbf{z}_{\mathbf{r}_n}{}^T\end{bmatrix}\right]^T$. Unfortunately, this is highly non-trivial in general, particularly for high-order multivariable systems, which greatly diminishes the utility of the formulation, (34). Moreover, it is noted that the interpretation of the linear system obtained by "freezing" the parameter θ in (34) at a particular value is unclear in the sense that no direct relationship exists between the solution to the frozen-parameter linear system and the solution to the nonlinear quasi-LPV system. Indeed, the frozen-parameter system associated with an equilibrium point is quite different from the conventional series expansion linearisation at that point. The related approach of Vidyasagar (1993) suffers from similar difficulties.

Output dependent quasi-LPV systems

Shamma & Athans (1992) consider nonlinear systems

$$\begin{bmatrix}\dot{\mathbf{y}}\\ \dot{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}\mathbf{f_y}(\mathbf{y})\\ \mathbf{f_x}(\mathbf{y})\end{bmatrix}+\begin{bmatrix}\mathbf{A_{yy}}(\mathbf{y}) & \mathbf{A_{yx}}(\mathbf{y})\\ \mathbf{A_{xy}}(\mathbf{y}) & \mathbf{A_{xx}}(\mathbf{y})\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \mathbf{x}\end{bmatrix}+\begin{bmatrix}\mathbf{B_y}\\ \mathbf{B_x}\end{bmatrix}\mathbf{r} \tag{63}$$

where $\mathbf{y}\in\Re^m$, $\mathbf{x}\in\Re^{n-m}$ and the input, $\mathbf{r}\in\Re^m$, has the same dimension as the subset of state, $\mathbf{y}$. It is assumed that $\mathbf{r}$ is uniformly zero at every equilibrium operating point and, in addition, it is assumed that the family of equilibrium operating points are smoothly parameterised by $\mathbf{y}$. Under these conditions, (63) may be transformed into the quasi-LPV system

$$\dot{\xi} = \mathbf{A}(\mathbf{y})\xi + \mathbf{B}(\mathbf{y})\mathbf{r} \tag{64}$$

where the matrices $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{y})$ are appropriately defined, ξ equals $[\mathbf{y} \quad \mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})]^T$ and $\mathbf{x_o}(\mathbf{y})$ is the equilibrium value of $\mathbf{x}$ corresponding to $\mathbf{y}$ (Shamma & Athans 1992).

Of course, few systems are of the form, (63): in particular, it is restrictive to require that the nonlinearity is dependent only on the quantity, $\mathbf{y}$, which parameterises the equilibrium operating points and that the input is uniformly zero at every equilibrium operating point. However, with regard to the former restriction, consider the nonlinear system

$$\dot{\mathbf{z}} = \mathbf{F}(\mathbf{z}, \mathbf{r}) \tag{65}$$

where $\mathbf{r} \in \Re^m$, $\mathbf{z} = [\mathbf{y} \quad \mathbf{x}]^T$ with $\mathbf{y}\in \Re^m$, $\mathbf{x} \in \Re^{n-m}$ and $\mathbf{F}(\cdot,\cdot) = [\mathbf{F_y}(\cdot,\cdot) \ \ \mathbf{F_x}(\cdot,\cdot)]^T$ is differentiable with bounded, Lipschitz continuous derivatives. (It should be noted that, when the output function $\mathbf{G}(\cdot,\cdot)$ does not depend on the input $\mathbf{r}$, the nonlinear system, (1), can be reformulated as in (65)). Assume that the family of equilibrium operating points are smoothly parameterised by $\mathbf{y}$: this condition determines the allowable partitionings of the state, $\mathbf{z}$, into components $\mathbf{x}$ and $\mathbf{y}$ but is otherwise quite weak. Adopting a similar approach to Shamma & Athans (1992) and employing a partial first-order series expansion about the equilibrium operating point, $(\mathbf{x_o}(\mathbf{y}),\mathbf{r_o}(\mathbf{y}))$, the nonlinear system may be reformulated as

$$\begin{bmatrix}\dot{\mathbf{y}}\\ \delta\dot{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \nabla_x\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))\\ \mathbf{0} & \nabla_x\mathbf{F}_x([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))-\nabla\mathbf{x_o}(\mathbf{y})\nabla_x\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \delta\mathbf{x}\end{bmatrix}+\begin{bmatrix}\nabla_\mathbf{r}\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))\\ \nabla_\mathbf{r}\mathbf{F}_x([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))-\nabla\mathbf{x_o}(\mathbf{y})\nabla_\mathbf{r}\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,\mathbf{r_o}(\mathbf{y}))\end{bmatrix}\delta\mathbf{r}+\begin{bmatrix}\varepsilon_y\\ \varepsilon_x\end{bmatrix} \tag{66}$$

$$\delta\mathbf{x} = \mathbf{x} - \mathbf{x_o}(\mathbf{y}),\ \delta\mathbf{r} = \mathbf{r} - \mathbf{r_o}(\mathbf{y}) \tag{67}$$

where

$$\varepsilon_\mathbf{y}=\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla_x\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}-\nabla_r\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r} \tag{68}$$

$$\begin{aligned}\varepsilon_\mathbf{x}=&\mathbf{F_x}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla\mathbf{x_o}(\mathbf{y})\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x}]^T, \mathbf{r}) - \nabla_x\mathbf{F_x}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}-\nabla_r\mathbf{F_x}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r}\\ &+\nabla\mathbf{x_o}(\mathbf{y})\nabla_x\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{x}+\nabla\mathbf{x_o}(\mathbf{y})\nabla_r\mathbf{F_y}([\mathbf{y}\ \ \mathbf{x_o}(\mathbf{y})]^T, \mathbf{r_o}(\mathbf{y}))\delta\mathbf{r}\end{aligned} \tag{69}$$

Assume that $\mathbf{r}$ is uniformly zero at every equilibrium operating point; that is, $\mathbf{r_o}(\mathbf{y})=0$ and $\delta\mathbf{r} = \mathbf{r}$. It follows that

$$\begin{bmatrix}\dot{\mathbf{y}}\\ \delta\dot{\mathbf{x}}\end{bmatrix}=\begin{bmatrix}\mathbf{0} & \nabla_x\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)\\ \mathbf{0} & \nabla_x\mathbf{F}_x([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)-\nabla\mathbf{x_o}(\mathbf{y})\nabla_x\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)\end{bmatrix}\begin{bmatrix}\mathbf{y}\\ \delta\mathbf{x}\end{bmatrix}+\begin{bmatrix}\nabla_\mathbf{r}\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)\\ \nabla_\mathbf{r}\mathbf{F}_x([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)-\nabla\mathbf{x_o}(\mathbf{y})\nabla_\mathbf{r}\mathbf{F}_y([\mathbf{y}\ \mathbf{x_o}(\mathbf{y})]^T,0)\end{bmatrix}\mathbf{r}+\begin{bmatrix}\varepsilon_y\\ \varepsilon_x\end{bmatrix} \tag{70}$$

Neglecting the perturbation terms $\varepsilon_\mathbf{y}$ and $\varepsilon_\mathbf{x}$, it is evident that (70) is of the quasi-LPV form (64). From Taylor series expansion theory, the perturbation terms $\varepsilon_\mathbf{y}$ and $\varepsilon_\mathbf{x}$ can be made arbitrarily small provided the magnitudes of $\mathbf{r}$ and $\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})$ are sufficiently small. Hence, provided $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ are sufficiently small, the solution to the nonlinear system, (65), is approximated by the solution to quasi-LPV system obtained by neglecting $\varepsilon_\mathbf{y}$ and $\varepsilon_\mathbf{x}$ in (70). It should be noted that the matrices, $\mathbf{A}(\mathbf{y})$ and $\mathbf{B}(\mathbf{y})$, in the quasi-LPV system do *not* correspond to the series expansion linearisations, about the family of equilibrium operating points, of the nonlinear system, (65).

The foregoing analysis relaxes the requirement that the nonlinearity is dependent solely on the quantity, $\mathbf{y}$, which parameterises the equilibrium operating points. However, this is achieved at the cost of restricting consideration to the class of inputs and initial conditions for which $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ are sufficiently small. Hence, any analysis based on this quasi-LPV formulation is strictly local in nature. It should be noted that, from (70), restricting $|\mathbf{r}|$ and $|\mathbf{x}\text{-}\mathbf{x_o}(\mathbf{y})|$ imposes an implicit constraint on the rate of variation of the states, $\mathbf{y}$ and $\mathbf{x}$. This constraint may, in general, be extremely restrictive since the region of validity can be vanishingly small.

Moreover, the requirement that $\mathbf{r}$ is uniformly zero at the equilibrium operating points is still necessary since it ensures that the input transformation, (67), associated with the series expansion is trivial and, in particular, does not vary with the equilibrium operating point. Indeed, this condition is central to both the approach of

Shamma & Athans (1992) and to the foregoing quasi-LPV reformulation. (When this condition is not satisfied, the quasi-LPV approximation is only accurate about the specific equilibrium operating point at which $\mathbf{r_o}$ is zero). In the specific control design situation where the nonlinear system to be reformulated is the plant and the controller output is the only input to the plant and the controller contains pure integral action, this condition can be satisfied by formally including the controller integral action within the plant so that the input, $\mathbf{r}$, to the augmented plant is zero in equilibrium. It should be noted, however, that in a more general context the requirement that the input is uniformly zero in equilibrium is rather restrictive; for example, when the input, $\mathbf{r}$, consists of command signals and/or disturbances.

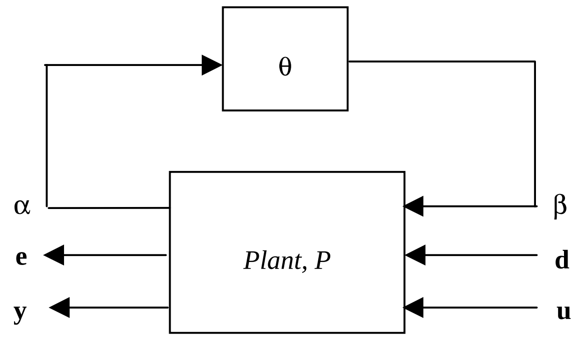

Figure 1 Parameter-dependent LFT system

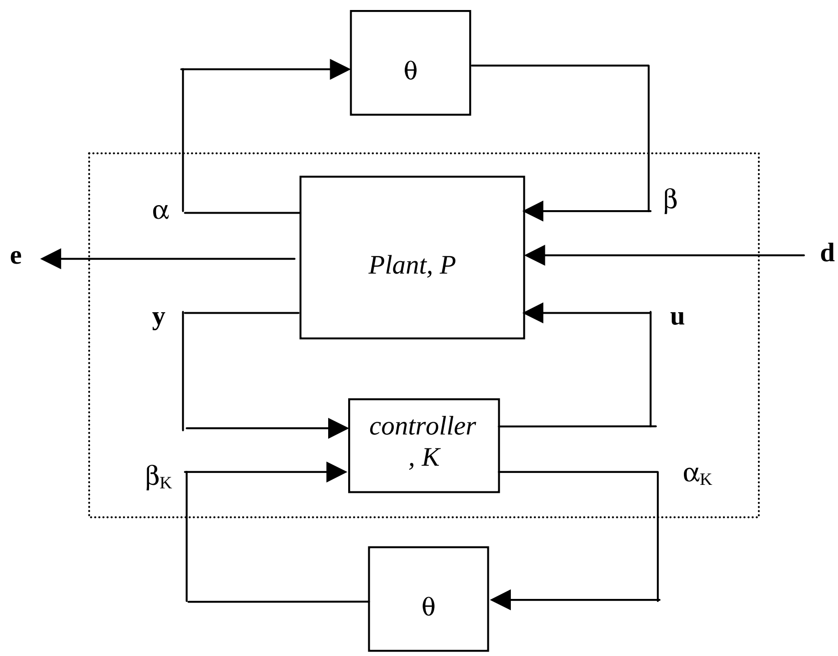

Figure 2 Closed-loop combination of parameter-dependent LFT plant and controller