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ADAPTIVE MPC BASED ON PROBABILISTIC BLACK-BOX INPUT-OUTPUT MODEL

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Abstract

A novel adaptive model-based predictive-control algorithm is proposed for the control of stable nonlinear dynamic systems. The model of controlled dynamical system is a Gaussian process (GP) model. It is a probabilistic, nonparametric, kernel, black-box model. An evolving GP-modelling method is used for the online model identification. Model-based predictive control utilising such a model results in an adaptive feedback control. The proposed control-method features are illustrated on a simulation example.

Key words: evolving Gaussian process model, model-based predictive control, adaptive control

1. Introduction. The paper presents a novel approach of model-based predictive control (MPC) [1] in which the model of controlled system is an Evolving system. The Evolving system is based on the probabilistic, nonparametric, kernel, Gaussian process (GP) model. The main idea of Evolving GP model [2] regarding the controlled system is the online adaptation to it, an input-output representation of it, and probabilistic prediction of its output signal. Nonparametric black-box modelling methods facilitate the modelling necessary for control design by avoiding acquiring knowledge about the first principles behind the dynamics of controlled system if it is time-consuming.

In order to model a dynamical system, the GP model is represented as a nonlinear autoregressive model with exogenous inputs (NARX) [3]. The corresponding GP model identification is based predominantly on the input-output

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signal measurements of the system to be controlled. The GP-based NARX predicts the output signal value in next time step as a normally distributed random variable that depends on delayed output and input signal values.

Online GP-model identification [2] follows the concept of evolving, i.e. self-developing systems. From the point of view of control systems, online model identification enables the MPC algorithm to be adaptive.

GP models for MPC control systems are described as input-output or state space. The survey of GP-model control methods [4] provides an insight into adaptive control with GP models.

The main contribution hereby is a novel nonlinear control of an input-output dynamic system with adaptive and probabilistic MPC algorithm employing the Evolving GP model. The closed-loop control performance will be demonstrated along with some variations in illustrative example.

2. GP model and its evolving upgrade for dynamical systems modelling. The Gaussian process [5] is an arbitrary collection of random variables $\mathbf{y}$ with joint normal distribution $p(\mathbf{y}) = \mathcal{N}(\mathbf{y}|\boldsymbol{\mu}, \mathbf{K})$ with mean vector $\boldsymbol{\mu}$ and covariance matrix $\mathbf{K}$. The GP is determined by the mean function $\mu(\mathbf{z})$ and the covariance function $k(\mathbf{z}_i, \mathbf{z}_j)$ which both define an arbitrary collection of joint random variables $\mathbf{y}$ from some input vectors $\mathbf{z}, \mathbf{z}_i, \mathbf{z}_j \in \mathbb{R}^D$. GP is suitable for regression modelling: the regressors of an input vector $\mathbf{z} \in \mathbb{R}^D$ affect the mean and covariance function value, meanwhile each regressand value, i.e. the output variable, is normally distributed with the mean and covariance at corresponding regressors value. Provided some measurements of the regressand, the application of the Bayes theorem tunes the model [5] according to the measurements. Therefore, the prior model mean and covariance is set to the posterior mean and covariance function in closed form. The posterior mean function $\mu'(\mathbf{z})$ and covariance function $k'(\mathbf{z}_i, \mathbf{z}_j)$ is given by:

$$\mu'(\mathbf{z}) = \mu(\mathbf{z}) + \mathbf{k}^{\mathrm{T}}(\mathbf{z}, \mathcal{Z}_0)\mathbf{K}_0^{-1}\mathbf{y}_0, \tag{1}$$

$$k'(\mathbf{z}_i, \mathbf{z}_j) = k(\mathbf{z}_i, \mathbf{z}_j) - \mathbf{k}^{\mathrm{T}}(\mathbf{z}_i, \mathcal{Z}_0)\mathbf{K}_0^{-1}\mathbf{k}(\mathbf{z}_j, \mathcal{Z}_0), \tag{2}$$

where $\mathbf{K}_0$ is the covariance matrix of modelled prior covariances between the corresponding measured regressands y_0, and $k(\mathbf{z}, \mathcal{Z}_0)$ is a vector of prior covariances between an arbitrary regressand $\mathbf{y}$ and the measured regressands $\mathbf{y}_o$ with corresponding regression vectors $\mathbf{z} \in \mathcal{Z}_0$. The prior mean is hereby assumed to zero, but the prior covariance function is additionally parametrized with a vector of hyperparameters $\boldsymbol{\theta}$. A posterior GP model with optimal hyperparameters is obtained with respect to the maximisation of the marginal likelihood $p(\mathbf{y}_0|\boldsymbol{\theta})$ or its logarithm [5]:

$$\log\left(p(\mathbf{y}_0|\boldsymbol{\theta})\right) = -\frac{1}{2}\log(|\mathbf{K_0}|) - \frac{1}{2}\mathbf{y_0}^{\mathrm{T}}\mathbf{K}_0^{-1}\mathbf{y_0} - \frac{n}{2}\log(2\pi), \tag{3}$$

where n is the number of measurements.

GP model is a kernel model and therefore fundamentally nonparametric as the inference of posterior mean and covariance function is analytically defined directly from data. The inference of posterior mean and covariance is computationally intensive as it requires the inverse of covariance matrix $\mathbf{K}_0$.

The problems of computational burden and numerical instability gave rise to sparse GP modelling methods [6] which try to represent the complete dataset $\mathcal{D}$ with a smaller set, also known as the active set $\mathcal{D}_A$, that retains as much information as possible. The Evolving GP model [6] is the case when the sparse GP modelling is configured for online application. It is inspired by Evolving systems which are self-developing systems as it may update online the hyperparameters $\boldsymbol{\theta}$, the active set $\mathcal{D}_A$, the choice of regressors inside $\mathbf{z}$, and it may also reconfigure the structure of prior covariance function of the GP model [6]. In this case, we restrict to the use of Evolving GP for updating the hyperparameters $\boldsymbol{\theta}$ and the active set $\mathcal{D}_A$ rather than updating all of the formerly listed. Let $e_r = (\mu'(\mathbf{z}_n) - y_n)$ be the residual of the new measurement y_n with corresponding $\mathbf{z}_n$, $\zeta > 0$ be a predefined residual threshold, $M_{\mathcal{D}_A} \in \mathbb{N}^+$ be the desired active set size limit, $f \in (0, 1]$ be the forgetting factor, $g(\mathbf{z})$ be the age of the regression vector $\mathbf{z}$. The active set and hyperparameters $\mathcal{D}_A, \boldsymbol{\theta}^*$ are updated online to $\mathcal{D}'_A, \boldsymbol{\theta}'^*$ under the following rules:

$$|e_r| > \zeta \implies \mathcal{D}'_A = \mathcal{D}_A \cup \{y_n, \mathbf{z}_n\} \implies \boldsymbol{\theta}'^* = \arg\max_{\boldsymbol{\theta}} \left[\log(p(\mathbf{y}_0|\boldsymbol{\theta}))\right], \tag{4}$$

$$\exists\{\mathbf{z}_m, y_m\} \in \mathcal{D}_A : k'(\mathbf{z}_m, \mathbf{z}_m) f^{g(\mathbf{z}_m)} \leq k'(\mathbf{z}_j, \mathbf{z}_j) f^{g(\mathbf{z}_j)}, \forall\{\mathbf{z}_j, y_j\} \in \mathcal{D}_A, \tag{5}$$

$$|\mathcal{D}_A| > M_{\mathcal{D}_A} \implies \mathcal{D}'_A = \mathcal{D}_A \setminus \{\mathbf{z}_m, y_m\}, \tag{6}$$

where the condition in (5) describes a linear independence test from [7] with additional weight term in a form of exponential forgetting factor.

GP modelling framework is also suitable for modelling dynamical systems. GP model may be used as nonlinear autoregressive model with exogenous inputs (NARX) where the regressors are taken from arbitrarily delayed input signal u and output signal y [3]:

$$y(k) = f\left(y(k-1), \ldots, y(k-l), u(k-1), \ldots, u(k-m)\right) + \epsilon(k), \tag{7}$$

where $\epsilon(k)$ is the output signal noise and k is the sampling time instant. All regressors at the time instant k are combined into a regression vector $\mathbf{z}(k) = [y(k-1), \ldots, y(k-l), u(k-1), \ldots, u(k-m)]^{\mathrm{T}}$. Note that $\mathbf{z}(k)$ may contain noise from the past regressand values which is discussed in [8]. In this case the signal noise is assumed to be not high enough to affect the training of GP model.

In deterministic NARX modelling the multi-step prediction is obtained iteratively with single-step prediction and so depends on deterministic past predictions, meanwhile the GP modelling framework enables the proper simulation approach to propagate the uncertainty of past predictions. The naive approach is to not propagate the uncertainty at all and take the expectation of predicted random variable instead.

3. Probabilistic adaptive model-predictive control. The Evolving GP model of the system to be controlled adapts itself with online identification to the operating region of the system. An optimisation algorithm finds the best control input signal candidate according to the criterion that reflects the desired closed-loop performance and depends on multi-step prediction of Evolving GP model. Such MPC algorithm is presented in Fig. 1, where w is the set-point signal, y is the output of the controlled system, and u is the control system input at the current time-step k. Let J be a general cost function $J = J\left(\widehat{y}(k+j), w(k), \Delta u(k+j)\right)$ that requires the predicted system output $\widehat{y}(k+j) \sim \mathcal{N}(\mu_y(k+j), \sigma_y^2(k+j))$, the corresponding control input difference $\Delta u(k+j) = u(k+j) - u(k+j-1)$, and the set-point signal value $w(k)$ for any $j \in \mathbb{N}^+$. Let the future control input values be defined from a parameter vector $\mathbf{p} = [p_1, \ldots, p_{H_c}]^{\mathrm{T}}$ as follows:

$$\widetilde{u}(k+j-1) = \begin{cases} p_j & \text{if } j \leq H_c \\ p_{Hc} & \text{otherwise} \end{cases}, \tag{8}$$

where H_c is the control horizon. The MPC control method solves the following unconstrained optimisation problem:

$$\mathbf{p}^* = \arg\min_{\mathbf{p}} J = \arg\min_{\mathbf{p}} \sum_{j=1}^{H_p} l\left(\widehat{y}(k+j), r(k,j), \Delta u(k+j)\right), \tag{9}$$

where $\mathbf{p}^*$ determines the optimal future control input signal $\widetilde{u}$, H_p is the prediction horizon, l is an arbitrary nonzero function, and $r(k,j) = (y(k) - w(k))e^{-j\frac{T_s}{T}} + w(k)$ is the reference trajectory. The optimisation problem in (9) is solved for each time-step k, and only the first value of optimal future control input sequence is applied. However, as the MPC model is probabilistic, the following case study will provide an illustration of different approaches involving the mean $\mu_y(k+j)$ and variance $\sigma_y^2(k+j)$ of predicted system output inside the cost function.

4. Case study. The proposed control method is demonstrated on a closed-loop simulation on an anaerobic wastewater treatment plant which is the Contois bioreactor model [9]. This bioreactor is a continuous, nonlinear, open-loop stable dynamical system of order two. The bioreactor input and output signals are

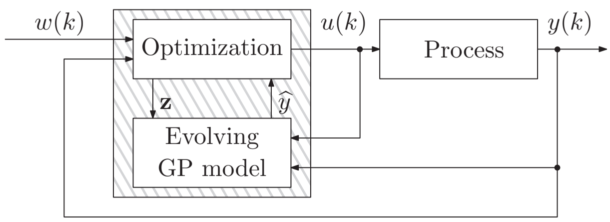

Fig. 1. The scheme of adaptive MPC algorithm

discretized with sampling time $T_s = 1$ hour. The input signal is the substrate concentration of the inflow $S_0(t)$ [$g_{\mathrm{COD}}\mathrm{dm}^{-3}$] and the output signal is the biomass concentration at the outflow of bioreactor $X = X(t)$ [$g_{\mathrm{VSS}}\mathrm{dm}^{-3}$]. Let the volume of the bioreactor $V = 5$ dm^3, the inflow rate of wastewater $Q = 1.49$ dm^{3}day^{-1}, the substrate concentration of outflow of the bioreactor $S = S(t)$ [$g_{\mathrm{COD}}\mathrm{dm}^{-3}$], the biomass concentration at the inflow side $X_0 = 0$ $g_{\mathrm{VSS}}\mathrm{dm}^{-3}$, the yield coefficient $Y = 0.2116$ $g_{\mathrm{VSS}}g_{\mathrm{COD}}^{-1}$, the death rate of biomass $k_d = 0.0131$ day^{-1}, the Contois kinetic parameter $B = 0.4818$ $g_{\mathrm{COD}}g_{\mathrm{VSS}}^{-1}$, the maximum population growth rate $\mu_{\max} = 0.9297$ day^{-1}, and the population growth rate $\mu = \mu(t)$ [day^{-1}]. The dynamical model of the bioreactor [9] is:

$$\frac{dS}{dt} = \frac{Q}{V}(S_0 - S(t)) - \frac{X(t)\mu(t)}{Y} \tag{10}$$

$$\frac{dX}{dt} = \frac{Q}{V}(X_0 - X(t)) + \mu(t)X(t) - k_d X(t) \tag{11}$$

$$\mu(t) = \mu_{\max}\frac{(S(t))}{(BX(t) + S(t))} \tag{12}$$

$$X_m(t) = X(t) + \epsilon,\ p(\epsilon) = \mathcal{N}(0, 0.001^2), \tag{13}$$

where $X_m(t)$ is the actual measurement of the biomass $X(t)$.

The control system is required to track the reference substrate concentration with notable convergence. The properties of closed-loop performance depend on setting both the identification process and the MPC parameters. The identification of GP model requires regression vector $\mathbf{z}(k) = [X_m(k-1), X_m(k-2), S_0(k-1), S_0(k-2)]^{\mathrm{T}}$ and the regressand $y(k) = X_m(k)$ composed from samples of signals. The prior GP mean is set to zero, the prior covariance is an exponentiated quadratic form with automatic relevance detection [5], the marginal likelihood is maximised with conjugate-gradients method limited to 20 iterations, and the design parameters for the online identification are $\zeta = 0.002$, $M_{\mathcal{D}_A} = 40$, $f = 0.975$. At the beginning of simulation, the bioreactor input is manually operated to collect $M_{\mathcal{D}_A}$ datapoints and to prepare an initial GP model for better computational stability. After day two, the bioreactor is set in proposed closed-loop control in which the GP model starts evolving.

The arbitrarily set parameters of the MPC control are the prediction horizon $H_p = 6$, the control horizon $H_c = 3$, and the reference trajectory constant $T = 2.4$ hours. Here we list four of possible variants of function l that determines the cost function J:

(a) the simple quadratic cost function, as described in [10], provides a closed-loop response that avoids highly uncertain regions and tracks the set-point signal:

$$l(r, y, \Delta u) = E\left[a^2||r - y||^2\right] = a^2||r - \mu_y||^2 + a^2\sigma_y^2,\ a = 1, \tag{14}$$

(b) the generalised quadratic cost is adopted from [4] and it contains an additional cost term of control-input signal and weight matrices $\mathbf{W}_{mu}, \mathbf{W}_{\sigma}, \mathbf{W}_{u}$. Such cost function may reduce the controller sensitivity to the output-signal noise and an arbitrarily smooth control-input signal, but it requires defining the weighting terms.

$$
\begin{aligned}
l(r, y, \Delta u) = ||r - \mu_y||^2_{\mathbf{W}_\mu} + ||\sigma_y||^2_{\mathbf{W}_\sigma} + ||\Delta u||^2_{\mathbf{W}_u}, \\
\mathbf{W}_\mu = 1, \mathbf{W}_\sigma = 100, \mathbf{W}_u = 1.
\end{aligned} \tag{15}
$$

Here, $||x||^2_{\mathbf{A}}$ denotes the squared weighted norm of a vector x, defined for some positive-definite matrix $\mathbf{A} \succ 0$, i.e. $||x||^2_{\mathbf{A}} = x^{\mathrm{T}} \mathbf{A} x$.

(c) the generalised quadratic cost is similar to (b), but it includes a bias term s for the acceleration of identification process with cautious excitation of the plant

$$
\begin{aligned}
l(r, y, \Delta u) = ||r - \mu_y||^2_{\mathbf{W}_\mu} + ||\sigma_y - s||^2_{\mathbf{W}_\sigma} + ||\Delta u||^2_{\mathbf{W}_u}, \\
\mathbf{W}_\mu = 1, \mathbf{W}_\sigma = 100, \mathbf{W}_u = 1, S = 0.006,
\end{aligned} \tag{16}
$$

(d) the saturating cost, described in [10], allows automatic exploration and exploitation. The control algorithm explores the operating region if the model is uncertain. On the other hand, if the model is precise, the cost function is more sensitive to the tracking error.

$$
\begin{aligned}
l(r, y, \Delta u) &= E\left[1 - \exp\left(-\tfrac{1}{2a^2}(y - r)^2\right)\right] \\
&= 1 - a\left(\sigma_y^2 + a^2\right)^{-\frac{1}{2}} \exp\left(-\tfrac{1}{2}\tfrac{(\mu_y - r)^2}{\sigma_y^2 + a^2}\right), a = 0.1.
\end{aligned} \tag{17}
$$

The resulting closed-loop responses for the given functions l in Fig. 2 illustrate the differences how the function l affects the smoothness and range of control signal input S_0, the value of predicted variance during the transition to an unknown operating region, and the tracking performance of the reference signal. In Fig. 2, the bioreactor output and the mean value of one-step-ahead model prediction are almost indistinguishable.

5. Conclusion. The proposed algorithm provides a possible solution for adaptive model-based predictive control of a nonlinear dynamic system, described with a probabilistic input-output model. An adaptive control algorithm allows adaptation of the controller to different operating regions of the controlled system, provided the transition between operating regions is sufficiently smooth. The inclusion of control input signal into the cost function is necessary for a closed-loop response with smooth control input signal. On the other hand, the addition of bias into the variance term in function (c) may affect the control algorithm to noninvasively explore the current operating region.

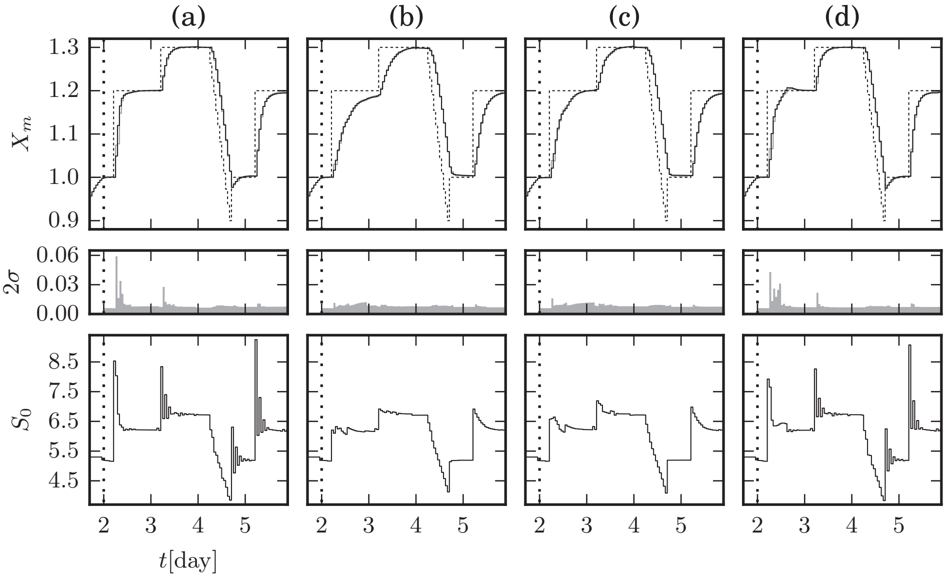

Fig. 2. Closed-loop responses of the proposed control method where each column is associated with the listed functions l. The upper row contains the set-point w in dashed black, the bioreactor output X in solid black, and the mean value of one-step-ahead model prediction in solid grey. A vertical dotted black line at $t = 2$ days indicates the transition from manual operation to the MPC control method. The middle row contains the double standard deviation of one-step ahead prediction. The bottom row shows the input signal S_0

GP model identification involves several computations in which the load increases with third power of the number of input data. The number of identification data is, however, kept small by using an online identification algorithm. A possible direction for future work is input-output model-based predictive control of open-loop unstable nonlinear systems.

REFERENCES

[1] Maciejowski J. M. (2002) Predictive Control with Constraints, Pearson Education.

[2] Petelin D., J. Kocijan, A. Grancharova (2011) Comptes Rendus de L'Academie Bulgare des Sciences, **64**, No 1, 117–124.

[3] Kocijan J., A. Girard, B. Banko, R. Murray-Smith (2005) Mathematical and Computer Modelling of Dynamical Systems, **11**, 411–424.

[4] Kocijan J. (2011) In: Proceedings of Special International Conference on Complex systems: Synergy of Control Communications and Computing, Ohrid, Republic of Macedonia.

[5] Rasmussen C. E., C. K. I. Williams (2006) Gaussian Processes for Machine Learning, Cambridge, MA, MIT Press.

[6] Petelin D., A. Grancharova, J. Kocijan (2013) Simulation Modelling Practice and Theory, **33**, 68–80.
[7] Chowdhary G., N. K. Üre, J. How, H. Kingravi (2012) In: Proc. of Conference on Guidance, Navigation and Control, Minneapolis.
[8] Quiñonero-Candela J. (2004) PhD thesis, Technical University of Denmark.
[9] Hu W. C., K. Thayanithy, C. F. Forster (2002) Process Biochemistry, **37**, 965–971.
[10] Deisenroth M., D. Fox, C. Rasmussen (2015) IEEE Transactions on Pattern Analysis and Machine Intelligence, **37**, 408–423.

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